Frequency as an extra dimension of Reality*

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Abstract

Over the last century, the quest to formulate physics to account for Reality has attracted a large number of theoreticians to propose various models that have tended toward a growing level of abstractness. While space and time have been largely recognized as the four fundamental dimensions that make our perceived reality, their completeness has been challenged either by positing hidden dimensions, or by exploring the possibility that spacetime itself is an emergent property of a more fundamental physical structure.

One quantity that has not entered this exploration is frequency—a measure of how often something repeats. Although it is featured in numerous physical contexts, it is normally implied that it is a mere parameter that is determined by the boundary conditions, or that it contains the same information as the time, period, wavelength, or energy—all supporting the notion that frequency is fully dependent on the other dimensions. In contrast, in psychophysics of vision, hearing, and touch, frequency is a quantity that appears independent, so that both its input and output are not directly dependent on the perception of space and time. Additionally, many important engineering applications treat frequency as a variable rather than as a parameter that is constrained alongside time.

This work explores the various conventions with respect to frequency in the physical, mathematical, and engineering literatures. It further scrutinizes frequency against the standard dimensions of space and time along nine properties that may be deemed universal. While a case for frequency being its own dimension can be made in different situations, a more general theorem is proven that states that only one of these three propositions can be simultaneously true:

1. Time is not a fundamental, obligatory dimension of Reality.
2. The universe is fully deterministic with total knowledge of past and future.
3. Frequency is a fundamental dimension of Reality.

The validity of this counterintuitive theorem is demonstrated using examples of epistemological nature with growing complexity and diminishing generality, which deal with problems of traffic flow, acoustic measurements, radio transmission, and psycholinguistics. It is proposed that if the incompatibility of the three propositions (or modes) is encountered within the analysis of any one system, there may be a discontinuity associated with the transition between modes. This is explored within the measurement problem and nonlocality of quantum mechanics, where it is suggested that these strange quantum effects may both be corollaries of the discontinuity between the modes of Reality. It is further proposed that the frequency dimension, should it exist, is nonlocal in some conditions and may have an ontological role within Reality, being neither in space nor in time. Dwelling on the interrelationship between determinism, time, and frequency, further metaphysical corollaries are explored in the appendices, including an emergent solution for the problem of foreknowledge, and by association, of the paradox of free will.

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> *"Now, what about induction? I referred to the fact that its premises rest on this notion of temporal invariance, which seems to be true only for the grossest of phenomena. Science has made great hay in that dimension, by creating laws which are essentially predictive laws of cyclical phenomena that define for us what part of reality goes around and comes around."* (McKenna, 1989)

1 Motivation and structure

Humans access the external reality through their senses, which function as arrays of detectors of various physical attributes of stimuli from the environment. Once the stimuli are peripherally detected, they are transduced to neural signals that can be perceived by the brain according to the specific stimulus within its *modality* (i.e., its corresponding sense, such as vision or hearing). The effects and implications of this indirect mediation of the external world on the resultant internally perceived reality have been under ongoing exploration over millennia within philosophy and science.

Of late, physics has been informally nominated as the branch of science that deals with the external Reality most rigorously and authoritatively, whereas perception and its internal reality has been variably dealt with within philosophy, psychology, biology, and neuroscience[1]. The interface between these domains is most directly addressed by psychophysics, itself a multidisciplinary science that does not claim to explain perception. Of these disciplines, physics has been most concerned with getting the closest to a true account of Reality that is unbiased by the particular layout and wiring of our senses.

As a major part of this exploration, a recurrent question has been whether the world geometry that is perceived as three dimensions of space plus one dimension of time is, in fact, nothing but a product of our perception. The alternative is that Reality is spanned by a different combination of dimensions, whose nature we may not be able to directly perceive or even conceive.

In its rush to uncover hidden dimensions of Reality, the physics foray has provided exciting ideas that challenge all naive perception of Reality, but may have neglected a more mundane contribution from psychophysics that repeatedly highlights the independence of *frequency* in sensation. This negligence is compounded by a tradition within physics of treating frequency as a de-facto parameter, in contrast with more applied fields where it is treated as a de-facto variable.

The present work sets to resolve the inconsistencies in the current science between various usages of frequency. While traditionally understood as a measure for how often a periodic phenomenon repeats, its present utility goes far beyond simple periodicity, to the point where the original meaning may be wanting in explaining its reach.

1.1 Outline

The logic of the text is as follows. It begins with a brief overview of the sensory and perceptual instantiation of the four accepted dimensions of Reality and contrasts them with that of frequency—generally a separate perceptual dimension that is perceived as many things that are not experienced as "how often" once they make it to consciousness (§ 2.1). A rough summary of the status of the four accepted dimensions and additional hypothetical ones in physics follows and a proposal that frequency may belong there is thrown in (§2.2).

The text continues through a semi-technical review of how the concept of frequency has evolved in physics, mathematics, signal analysis, and engineering ever since its original definition as the reciprocal of the period appeared some 400 years ago (§3). Although it has not been presented in this way before, the basic elements of the material in this section should be familiar to many in the physics, mathematics, and engineering disciplines and is the bread and butter of countless others. As the degree of complexity of the systems studied increases, it is shown how the frequency concept has drifted away from its restricted original use, and has become much more sophisticated and rich, but also problematic in some cases. The latter subsections in this technical review emphasize paradoxes and limitations of the frequency concept and contrast them with the original, static definition of

[1] In the following, external *Reality*, should it exist, is capitalized, whereas its image—the observed, sensed, perceived, or subjectively experienced *reality*—is lowercase. However, the distinction between the two is not always going to be clear cut. This nomenclature may serve to acknowledge the Kantian notion of a *thing-in-itself*, which, although controversial, is unavoidable when mixing physical and psychophysical points of view: "*...[T]hings as objects of our senses existing outside us are given, but we know nothing of what they may be in themselves, knowing only their appearances, i.e., the representations which they cause in us by affecting our senses.*" (Kant, 1783 / 2001, §13, Remark II).

frequency. The conclusion of this review is that the standard definition of frequency as a parameter does not capture its realistic and logical use in many practical cases. A more universal definition for frequency is offered (Definition 1).

Section §4 attempts to distill universal properties of the physical and perceptual dimensions of space and time. Frequency is contrasted with each property and its hypothetical elevation to the level of dimension is considered. The synthesis of these ideas results in the theorem that appeared also in the abstract, which entails that only one out of three possible modes of Reality holds at any given moment (§5). The epistemological validity of the theorem is initially illustrated using four examples: a toy model of road traffic, bioacoustic field measurements, radio communication channel design, and psycholinguistic homonym processing models (§6). It is seen that separating the three modes of Reality in real-world cases is often messy. It is further hypothesized that switching between modes may lead to a discontinuity of an unknown nature. These ideas are examined in two separate ontological examples from quantum mechanics. In the measurement problem, the deterministic–probabilistic incompatibility between the pre-measurement and post-measurement quantum states has been a famous conceptual stumbling block in its interpretation. It is argued in §7 that this incompatibility may be inevitable. The second example relates to quantum entanglement and nonlocality—a bizarre and yet well-established physical phenomenon—only to be repeatedly challenged as stemming from an unrealistic and nonphysical interpretation of the results and the formalism. Here, the idea of nonlocality is embraced as an inherent feature of the frequency dimension, which is suggested to give rise to five-dimensional objects, under entanglement (§8).

Challenges to the premise of the main ideas of the paper, as well as several other topics and connections to other topics, are discussed in §9 and briefly concluded in §10.

Finally, in the first appendix (§A), there is a separate treatment of the version of determinism that is used in this work, which is tied to the existence of the Fourier integral. The last appendix on metaphysics provides several alternative formulations to the theorem, from which a solution to the foreknowledge and free-will paradox appears to emerge (§B).

1.2 About the text

This text is written in academic language and uses jargon(s) that may be difficult in parts to parse for some readers, because of the multitude of disciplines that are blended in. Where applicable, I attempted to provide brief definitions, references, explanations, and illustrations that clarify the concepts for readers that are not directly acquainted with this or that discipline. Several subtler definitions, historical notes, and terminological comments have been confined to in-page footnotes, which would have been more disruptive to the reading flow had they been incorporated in the main text, but may still hold valuable information (notably, Footnotes 1, 11, and 12).

Overall, this is by no means an introductory-level text nor a complete review of any of the topics touched upon. A portion of this work assorts a large number of equations and formulas that would be familiar to the majority of readers in the natural sciences and engineering, but may be quite impenetrable to many others. For the most part, these equations are presented without derivation, while no new expressions are introduced that do not appear elsewhere. The formulas serve as anchors to abstract concepts that are in prevalent use in the sciences and mathematics, whose meaning and logic transcends the equations themselves. That said, I am positive that it can nevertheless be intimidating for the less mathematically-inclined readers, whom I therefore encourage to read around the equations, focus on the logic, and try to gather the main points from the illustrations and interim discussions. All this holds also for the choice of examples throughout the second half of this work, which may appear arcane and technical, depending on the particular expertise of the reader. In this sense and others this work cannot be readily classified into any one particular discipline, although it weighs strongly to physics, philosophy, perception, psychophysics, neuroscience, engineering, signal processing, statistics, harmonic analysis, and ultimately, metaphysics.

A final remark would be in place about the philosophy of this work that brings together what appears that should belong to either psychophysics or physics. If the distinction between the two appears to be blurred, it is because it indeed tends to be blurred. The approach taken here is reminiscent of John von Neumann's version of "*the principle of psycho-physical parallelism*", which he invoked in his interpretation of quantum mechanics (von Neumann, 1932 / 2018, p. 272): "*...it must be possible so to describe the extra-physical process of subjective perception as if it were in the reality of the physical world; i.e., to assign to its parts equivalent physical processes in the objective environment, in ordinary space.*" Hence, until the point of conscious perception itself begins—a rather vague thing that is not clearly localizable to any one point or area in the brain and may not be amenable to our analytical treatment—any intermediary physical process, including all biological signal transmission whose exact nature we may not fully understand, must still follow all otherwise-universal laws of signals, waves, statistics, communication, or any other analytical frameworks that are regularly employed outside of the biological machinery.

2 Introduction

2.1 Geometrical and temporal detection in sensation and perception

The starting point for any discussion about reality is our sensory apparatus and the ensuing perception, which mediate all observations of the external world. The different senses tend to provide complementary detection of different body parts, so that every part is covered by at least one or two senses (Fig. 1). Several senses are particularly well-equipped to deal with remote stimuli that are external to the body and are carried by radiation (hearing, vision, heat, magnetoreception) or by changes in concentration of certain chemical compounds (olfaction).

The geometrical arrangement and relevance of the senses are also found within the brain. The brain contains topographical map representations that reflect the peripheral structure of several senses, such as *somatopy* in touch that maps the skin (Penfield and Boldrey, 1937; Flanders, 2005), *retinotopy* in vision that corresponds to the optical image on the retina (Fishman, 1997; Wandell et al., 2007), *tonotopy* in hearing that corresponds to cochlear place (Pickles, 2012; Triarhou and Verina, 2016; Ruben, 2020), and an odor map that corresponds to primitive dimensions of olfaction (Lee et al., 2023). While these maps tend to be distorted at the cortical level (Flanders, 2005), they provide a perceptual gateway for mapping of the environment, inasmuch as its geometry is causally reflected in the peripheral sensory response (Uchimura et al., 2024). With additional processing of the neural signal, as well as with cross-modal *binding* of the sensory information that is processed as belonging to the same object (Roskies, 1999), the various senses provide information to the perceiver about his/her own position and how he/she is localized with respect to various objects within the environment (e.g. Lackner and DiZio, 2005). Bound spatial perception also includes information about the orientation of the perceiver's own body and possible interactions with external objects (Fig. 1, left). The integrated information about the world from the senses is positioned in a three-dimensional (3D) perceptual geometry that includes the sensing individual, who occupies part of that 3D space and serves as an internal reference.

In addition to the spatial information that is provided by the senses, information regarding temporal changes in the sensory stimuli can also be extracted from the detected signals, which can in turn be used to produce the time perception of the individual within the cortex (Allman et al., 2014). However, while different senses have different temporal precision associated with their stimulus duration judgments, time perception is not associated with a specific modality and is not anchored to a dedicated sense organ (Grondin, 2010; Allman et al., 2014). Hence, time perception may be thought of as a supra-modal sense that is central to the entire perceptual system (Fig. 1, middle).

Despite the difficulty in understanding how perception manipulates or reduces the high mathematical dimensionality of natural stimuli (e.g., Shepard, 1994; Auffarth, 2013; Welchman, 2016; McAdams, 2019; and somewhat indirectly, Chen et al., 2022), all percepts that originate in the sensory apparatus correspond to certain physical properties of the stimulus and its environment that may not be encapsulated solely in spatial and temporal information[2]. As such, perhaps the most common additional attribute of stimuli in several modalities is their frequency content, or their (power) spectrum. In humans, vision (Mollon, 2003), hearing (Fletcher, 1940), touch (Talbot et al., 1968; Johansson et al., 1982; Bolanowski Jr et al., 1988), and balance (Todd et al., 2008) all produce perceptions that are causally linked to the stimulus frequency, which is detected by appropriately tuned sensory receptors[3]. In some cases, the receptor tuning has been explicitly analogized to frequency channels that are locally demodulated to obtain baseband (i.e., low-frequency) signals

[2]For simplicity, we omit from the discussion certain stimuli that are artificially generated and, in some cases, can be designed to "fool" the ecologically evolved correspondence to natural stimuli (e.g., using video displays or loudspeakers), so that their perception rarely corresponds to objects encountered in the natural environment.

[3]According to an interesting hypothesis, olfaction too may generate differentiated sensations of odorants determined by their vibrational spectra, which in molecular (Raman) spectroscopy are known to be unique in a particular infrared fingerprint region (Malcolm Dyson, 1938; Wright, 1977; Turin, 1996). However, under psychophysical and physiological scrutiny, this hypothesis has so far failed to receive experimental support (Keller and Vosshall, 2004; Block et al., 2015).

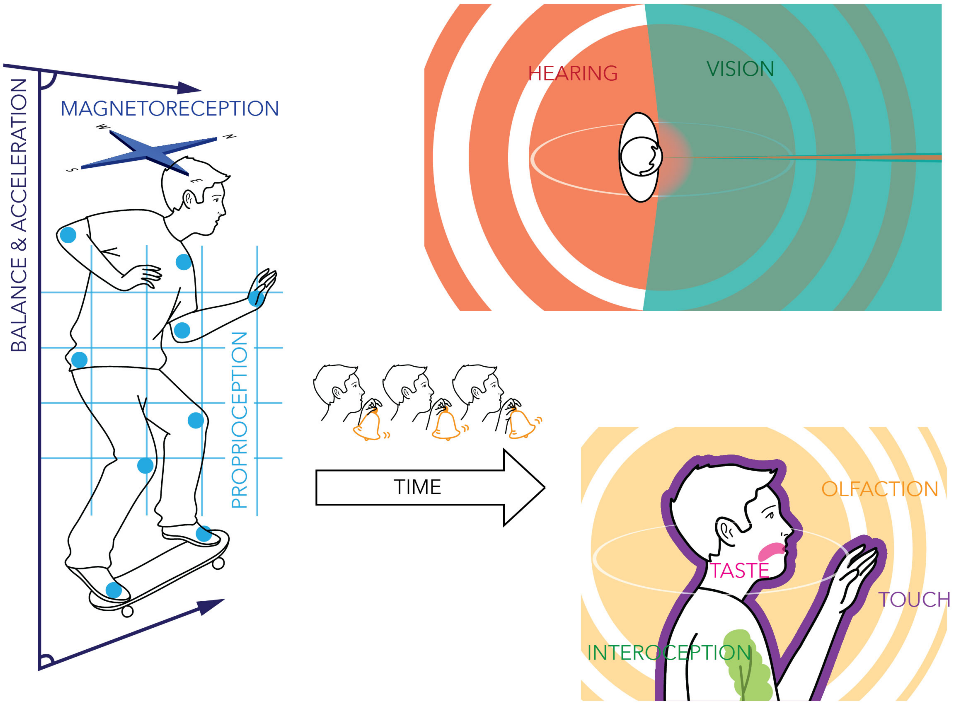

Figure 1: The sensory mapping of the physical environment is delegated to the different senses that are located in strategic places in the body, which are suitable to detect either direct-contact or remote stimuli. Most body parts are covered by at least one sense (e.g,. touch, chemesthesis, or interoception) with key areas that interface the environment covered by more than one sense. (Original illustration by Jody Ghani.)

that vary slowly in time and space, at rates that can be directly coded by neuronal spiking (e.g., Rhodes, 1953; Weisser, 2021, pp. 122–123). While different peripheral channels can interact and be segregated or fused in perception, they are understood as providing information that may not be available in the stimulus spatial and temporal attributes alone, as can be gathered, for example, from the effects of color on object recognition (Wurm et al., 1993)[4], or speech recognition when some parts of the spectrum are filtered out (French and Steinberg, 1947; Kasturi et al., 2002). Thus, many of these percepts are experienced as something (*qualia*) that does not seem to directly correspond to the question of “How often” that is technically associated with the concept of frequency. Rather, the spectral or spectral-equivalent perceptual representation of the stimulus provides a “What” kind of information about the physical object that generates the stimulus[5], which is arguably mandatory in all senses that rely on physical waves from the environment[6].

It therefore appears plausible that the perceptual experience of the external reality has to include, at the very least, five dimensions (5D) with frequency, spectrum, or channel(s) being the extra fifth dimension on top of the usual 4D space and time.

2.2 Dimensions of Reality within physics

While space has always played an obvious role in physics, the inclusion of time in the standard dimensional count is a relatively recent development that was realized only with the advent of Einstein’s special theory of relativity. In fact, both D’Alembert and Lagrange had already proposed that time should count as a fourth dimension, a century and a half before Minkowski formalized the concept of *spacetime* (Minkowski, 1908; Cajori, 1926). And while the very intangible nature of time is in

[4]Note that even in monochromatic vision as is achieved by the *rod* (“black and white”) *photoreceptors* as is common in some animals, vision is still spectrally tuned, only to less narrow frequency channels than the *cone* (color) *photoreceptors*.

[5]The distinction between “*Where*” and “*What*” types of processing has been suggested as a fundamental organizing principle of the brain in vision, known as the *dual stream model* (Trevarthen, 1968; Schneider, 1969). This model was later expanded for hearing as well. See (Weisser, 2021, p. 40) for further references.

[6]Spectral band-limitation is well-ingrained in modern sensation and perception science, and yet frequency analysis as a general property of sensory channels has not been generalized beyond the specific modalities and no general reviews are available within the sensation and perception literature, to the best of my knowledge.

itself opaque, its inclusion as a an inseparable aspect of space has conceptually opened the door for even wilder theoretical proposals of additional dimensions of Reality that go beyond the perceptually observable space. These have provided attractive avenues in the attempts to unify relativity, electromagnetic, and quantum theories (beginning from Nordström, 1914; Kaluza, 1921; Klein, 1926; for more recent references see Gonzalez-Ayala et al., 2016). A highly influential conjecture has been the *anti de-Sitter / conformal field theory (AdS/CFT) correspondence*, which relates a five-dimensional gravitational theory to a four-dimensional quantum field theory without gravity, showing how all the information in the volume of the former can be projected onto a four-dimensional surface of the latter, all being part of a ten- or higher-dimensional universe (Maldacena, 1999). However, the AdS space assumes a negative scalar curvature, which requires a negative cosmological constant, whereas the observed cosmological constant of the universe is positive, for which a de-Sitter space may rather be appropriate.

Common to the various higher-dimensional theories has been the understanding that any such extra dimensions ultimately have to be reducible to the measurable four dimensions that are accessible to our senses (e.g., Green et al., 1987, p. 15). The extra dimensions in such theories are then either mathematically constructed, or assumed to map to very small geometries that are curled and topologically compact and are not amenable to observation using current measurement methods. According to one theoretical analysis, spacetime that is specifically four dimensional has special properties that enable life and physics as we know it (Tegmark, 1997). Additional mathematical features that are unique to four-dimensional geometry can be considered particularly attractive in physical modeling (Woit, 2022).

Insights from such physical theories of extra dimensions, along with the realization of how difficult it may be to formulate a consistent physical theory of extreme spatial and temporal scales, have influenced some ideas regarding the validity of perception itself. In its most sensational form, it has been hypothesized that the four-dimensional reality emerges on a macroscopic scale from a higher-dimensional space that exists at scales that are too small to be measured—something that may entail "*doom*" on the spacetime dimensional reality as we naively perceive it (Witten, 1996; Cole, 1999; Arkani-Hamed, 2014)[7]. In turn, this has led some scholars to suggest that perception may deliver an image of Reality that is altogether divorced from the "actual" Reality and is only geared to satisfy the evolutionary needs of the organism (Hoffman et al., 2015; cf. Footnote 1).

2.3 The absence of frequency from the physical dimensional count

In the quest to account for both observable and hidden physical dimensions, the essential frequency variable of key sensory systems (§ 2.1) has been left out of all discussions within the physics and philosophy literatures, perhaps with the exception of a mention in an overlooked proposal by Wiener and Struik (1928).

Why is it that the 4D spacetime has become the de-facto standard in physical representation without considering frequency as an extra dimension? While it is only possible to speculate here, at least three explanations come to mind. First, all physicists are trained in Fourier analysis—mainly in the context of solving partial differential equations. In the modern presentation of this technique, time and frequency appear as *reciprocal domains* that essentially contain the same information about the system (e.g., Sommerfeld, 1949, p. 21), which suggests that including frequency as a dimension would be superfluous, given that it is completely dependent on time. Second, unlike space, frequency does not refer to anything that is intuitively or immediately tangible—even less so than time, which has already suffered from this issue (Cajori, 1926). Third, frequency is a much more modern concept than time. The concept of frequency was introduced (not by name) only in 1585 by Giovanni Battista Benedetti, picked up by Galileo Galilei, and refined over the subsequent centuries (Dostrovsky, 1975)[8]. Filtering, which constitutes the most fundamental operation in the

[7]Historically, the sense of impending dimensional doom was originally spelled in the translation of Minkowski (1908, p. 75) at the opening to his seminal lecture: "*Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*" The original German text is only slightly less evocative: "*Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken* [completely decaying into shadows] *und nur noch eine Art Union der beiden soll Selbständigkeit bewahren.*" In the subsequent mention (p. 80, Ibid) the word "completely" ("*völlig*") is no longer there, so the translators stuck to "*fade away into shadows.*"

[8]In his analysis of the cause of simple consonances of musical intervals, Benedetti proposed that they occur when there is a simple integer ratio between the numbers of vibrations ("*percussions*"; in Latin, "*percuffiones*") of sound. In the example he gives, the sound is produced by two parts of a string that is divided by a bridge, and are inversely proportional to the string part lengths (Benedetti, 1585, p. 283). See partial translations from Latin in Palisca (1985, pp. 258–261), Palisca (1994, pp. 214–215), and further discussion in Capecchi and Capecchi (2023).

spectral domain that one can perform on a broadband time signal, is an even more recent concept that was formally invented for electric circuits only a century ago by Campbell (1917)[9]. Nowadays, filters are ubiquitous and any electronic detector can be associated with a filter that effectively limits its applicable spectral range (either deliberately designed or imposed by parasitic elements in the system), but the signal processing theory that facilitates this understanding is barely a century old (Campbell, 1922)[10]. Signal processing theory had anyway matured well after the dominant physical theories of the day had become established.

While these explanations may not fully capture the absence of frequency from the standard dimensional discussion, they highlight the problems involved in attempting to find out whether frequency may be, in fact, an independent variable and dimension of Reality.

2.4 The present exploration of frequency as a dimension

The present work outlook is motivated by the dimensions of Reality as are phenomenologically perceived by our senses, rather than with hidden dimensions that the senses may or may not be privy to. As such, it focuses on frequency as is conventionally conceptualized and materialized in the sciences and challenges its current non-dimensional status. The conditions are explored as for whether frequency is:

1. A parameter[11].
2. A variable that carries the same information as time, may be derived from it, and as such, equivalent to it.
3. An independent variable that may also vary in time.
4. A dimension of Reality that is distinct from both time and space.

Each option is categorically escalated compared to the previous. The relevance of options 1–3 can be explored using deduction alone based on first principles. Section §3 deals with 1–3, by referring to the quintessential appearances of frequency in physics, mathematics, and engineering. Option 4 that entails that frequency, which is neither a parameter nor dependent only on time, can be considered its own dimension is tested against several universal features of the standard space and time dimensions in §4. The conditions for frequency becoming its own dimension are then synthesized into Theorem 1 in §5.

3 Evolved approaches to frequency

This section assorts essential and defining occurrences of frequency within physics, engineering, and mathematics. As no new physics or mathematics is presented here, some readers may find certain elements of this review to be overly basic. However, the novelty here is in the uncovering of an otherwise subtextual narrative relating how the concept of frequency has evolved well beyond its initial usage and original definition. It is shown how the different definitions or usages of frequency either contain additional supra-dimensional parameters (i.e., that are not temporal or spatial), or they require an ultra-deterministic conception of Reality. As such, the first part of this narrative serves as a survey of many of the familiar concepts in basic introductions to oscillatory phenomena, but with emphasis on definitional intricacies, paradoxes, and contradictions that had not been previously all put together in writing. The latter part of the review (from §3.5) focuses on complementary approaches to frequency within modern time–frequency analysis that may be less familiar to some of the readers.

[9] Alexander Graham Bell (1875) described a mechanical method to separate transmissions of different frequencies, which can arguably count as a primitive approach to bandpass filtering and would be then the first one in print.

[10] To the best of my knowledge, there is no rigorous historical account of early signal processing theory that preceded the digital age (Oppenheim and Schafer, 2009, pp. 5–8). George Ashley Campbell (1922) wrote the first publication that formalized filter theory, following his very own patent of the first electrical filter (Campbell, 1917), and as such seems to be an appropriate cornerstone to designate the beginning of analog signal processing theory.

[11] In this context of frequency, a *parameter* is a fixed quantity of the system, which is distinct from a universal constant, and distinct from a variable that is manifestly changing. This is different from the idea of *time as a parameter*, which will be encountered in two varieties later in the text. First, time in some dynamical systems is considered parametric if it does not explicitly appear in the Hamiltonian or the dynamical differential equation (i.e., when the differential equation is autonomous). Second, parametric time is often used only as a measure of duration—a “time ruler” of a sort—which is distinct from time itself.

Frequency has been incorporated into two main modeling approaches to dynamical systems. The historical one is wholly *deterministic*[12]. It provides analytical (closed-form) solutions and intuition and has been traditionally used to introduce these topics in fundamental physics and engineering courses. Its main results have been studied using different families of differential equations and their solutions have given rise to powerful analytical techniques that carried over to various implementations in both analog and digital signal processing. Similar results keep on appearing in different guises in modern physics and engineering, so the relevance of this perspective has not waned. The second, statistical approach emerged more recently and is based on the analysis of signals whose particular instantiation is either unknown, unknowable, or unimportant, whereas analysis of their ensemble properties provides robust information. Ultimately, both the *deterministic* and the statistical approaches to frequency should account for the same physics, but they provide and rely on different types of intuition and information about the phenomena at hand.

As the physical and mathematical understanding of dynamical systems has become more sophisticated, so did the concept of frequency has been gradually expanded to include a wider array of conditions that did not originally lend themselves to *spectral* (i.e., frequency) analysis (Fig. 2). Beginning from oscillatory systems in complete equilibrium, frequency was incorporated in the description of systems with multiple modes of motion, in waves, in the description of lossy systems, and in accounting for the effects of external forces. Then, a profound conceptual jump has been to have frequency available for the description of aperiodic oscillations, nonlinear systems, and with arbitrary forces driving the system, for which periodicity is, at best, local. In dealing with the latter systems, it is impractical to speak of a time-independent frequency, although the classical definition of frequency that is time independent may be applied notwithstanding and may then lead to a description that is mathematically correct, but is of little practical use.

Throughout this analysis we refer somewhat interchangeably to waves, signals, oscillations, vibrations, periodic and cyclic motions, disturbances, fields, and stimuli. While these terms do not mean the same thing, their mathematical formulation and usages within the sciences that pertain to the present context is more similar than not. Thus, a "signal" here is taken as a general function of time and frequency, whose spatial dependence is of secondary importance. The information that the signal carries—its specific message—is immaterial, except for the fact that it can be seen as meaningful for a conscious receiver, in certain situations. Without loss of generality, it should be understood that the signal can take the form of an arbitrary waveform, time function, time series, variable, measurement, stimulus, etc. This enables us to make freer use of generic mathematical concepts developed within harmonic analysis or signal processing theory that may be universally applicable.

3.1 Ideal oscillators: frequency as a parameter

3.1.1 The simple harmonic oscillator

The *frequency* f (also called *rate*) of a periodic motion is defined as the reciprocal of its *period* T (also called *cycle*)—a fixed duration that characterizes the motion and provides the information about how often it repeats

$$f = \frac{1}{T} \tag{1}$$

[12]The adjective "deterministic" is used in this work in two related but subtly different meanings, which are both common in different literatures, but may give rise to no small amount of confusion due to equivocation. In quantitative analysis, "*deterministic*" (appearing in *slanted font* in the text) indicates that everything being modeled appears with certainty (probability equal to one). If a *deterministic* system can be modeled, it strictly relies on physical laws that are mathematically expressible in closed-form formulas (often based on differential equations), whose solutions should be precisely reproducible. Any statistical or probabilistic aspect of these models is generally implicit and empirical, rather than inherent to the model. In contrast, "deterministic" (normal font in the text) refers to the philosophical and metaphysical property of determinism—the quality of the remote past and future being completely determined by the present state, given law-governed Reality (see the Laplacian definition of determinism in §A.1). Despite their close relation, one usage of the word does not necessarily entail the other. Most importantly, we will argue that some *deterministic* systems do not entail determinism (mainly in §A.2 and §7.2). The two meanings converge in the present text at places where we utilize the existence of the Fourier spectrum as an indicator of determinism. In these cases, *deterministic* modeling and determinism lead to something happening with probability equal to one, deterministically. The equivocation is all but gone in the antonymous case: an *indeterministic* system and model entail indeterministic knowledge and, hence, Laplacian indeterminism applies to it as well, locally.

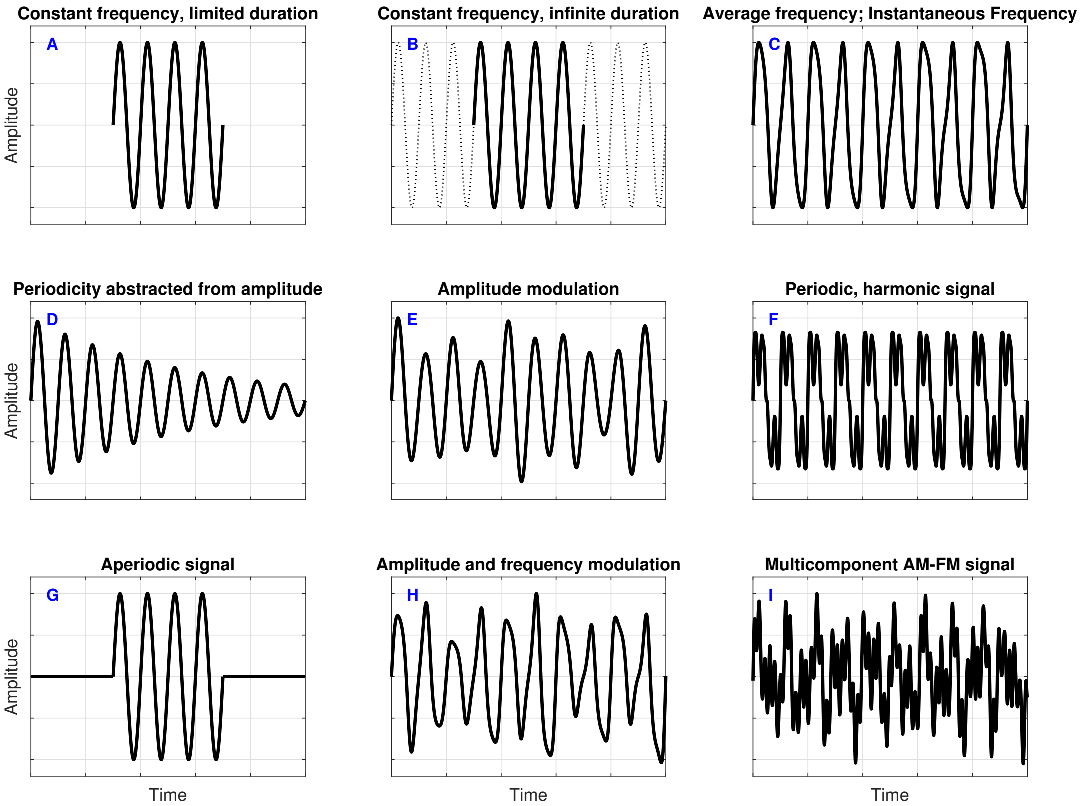

Figure 2: Visualization of the escalation of the concept of frequency with time signals of increasing complexity. **A**. Frequency is a parameter that is calculated from the limited duration measurement of simple periodic motion and is undefined outside of the observed interval. **B**. Frequency is a parameter, whose instantiation implies constant (inertial) periodic motion in the remote past and future. **C**. Frequency is an average of imperfect periods, either due to the measurement errors or due to instability in the oscillation. The average, nevertheless, produces the same long-term periodic motion as the simple harmonic oscillator. A different way to relate to this waveform is to define a time-dependent, instantaneous frequency. **D**. When loss of energy (damping) is included, a strict definition of periodic motion would emphasize that each period is slightly different in amplitude than its neighbors, so periodicity is not met here. Nevertheless, using a small correction, a constant frequency can still describe the oscillatory part of the motion, separate from a damping term that takes care of the decreasing amplitude (Eq. 25.) **E**. A periodic force of slower frequency than the oscillation drives the system, or modulates the signal in amplitude, further blurring the aspect of simple harmonic motion periodicity. **F**. Complex (non-sinusoidal) signals that are still periodic can be modeled using a Fourier series decomposition of the waveform to a sum of sinusoids whose frequencies are integer multiples (harmonics) of the fundamental frequency. As in B, the periodicity is extended to the remote past and future. **G**. An extension of the Fourier-series period to infinity produces the Fourier transform, which allows for modeling of aperiodic signals that comprise a continuum of sinusoids with parametric frequencies. **H**. Combining the slow variations in amplitude (E) with the variations in frequency (C) gives rise to a so-called AM–FM signal, whose only constant may be the average (center) frequency. **I**. A broadband signal can be decomposed to many narrowband AM–FM signals of the form of H. This form produces complex waveforms that do not necessarily disclose a clear periodicity, unless the components are separated using band-pass filtering, or another form of frequency tracking.

The frequency is measured in reciprocal time units s^{-1} or Hertz (Hz). Eq. 1 refers, for example, to the period of an idealized small-amplitude mass–spring system that can be computed from

$$T = 2\pi\sqrt{\frac{m}{s}} \tag{2}$$

with m the mass and s the stiffness of the spring—both of which can be estimated from static mechanical measurements. Similar dynamics accounts for the small-angle periodic movement of the

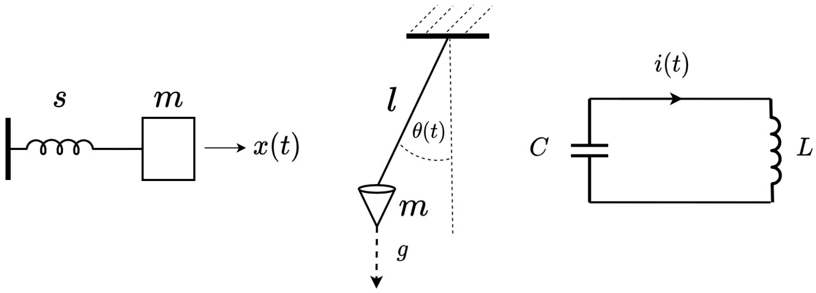

Figure 3: Three simple harmonic oscillators: mass–spring (left), pendulum (middle), inductor–capacitor (LC) circuit (right). All three systems are described by the same ordinary differential equation (5), where frequency is a parameter defined as the reciprocal of the period, which is itself determined by the various constants of the system.

pendulum, where

$$T = 2\pi\sqrt{\frac{g}{l}} \tag{3}$$

with g being the gravity of Earth and l is the length of the pendulum rod. Another basic system—an ideal LC (inductor–capacitor) circuit resonator has the period of

$$T = 2\pi\sqrt{LC} \tag{4}$$

with L being the inductance and C the capacitance. The three systems, sketched in Fig. 3, are examples of *simple harmonic oscillators* that are described by the same ordinary differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{5}$$

where x is displacement in the mass–spring system, angle of the pendulum, or electric current in the LC circuit, in our examples. The solution to the simple harmonic oscillator is then given by

$$x(t) = A\cos(\omega t - \varphi) \tag{6}$$

where the *angular frequency* $\omega = 2\pi f$ conveniently expresses the frequency in units of phase (radians per second). The *amplitude* A and the *phase* φ are parameters of motion determined by the initial conditions on x and its first derivative dx/dt (e.g., the initial displacement and velocity, or current and its first time derivative) at an arbitrary time t_0 (see Fig. 4). The frequency is usually referred to as the *resonance* of the system. Two more equivalent forms exist of the solution of Eq. 6—either as a sum of a sine and a cosine

$$x(t) = a_1\sin(\omega t) + b_1\cos(\omega t) \tag{7}$$

where a_1 and b_1 are real amplitudes, or alternatively,

$$x(t) = \mathrm{Re}\left[c_1 e^{-i\omega t}\right] \tag{8}$$

with c_1 being a *complex amplitude* and $i = \sqrt{-1}$, taking for $x(t)$, by convention, the real value of the complex exponential[13]. The constants a_1 and b_1 or c_1 are determined by the initial conditions. It is worth noting that because Eq. 5 does not depend on time explicitly (technically, an *autonomous* differential equation), it is invariant under translation in time—only the time elapsed $t - t_0$ matters between the time t_0 at the initial condition and the present time t, rather than their absolute values (Kloeden and Rasmussen, 2011).

The simple harmonic oscillation is plotted in Fig. 2 B. Despite its simplicity, ever since its introduction by Euler (1750 / 2021), the harmonic oscillator model has provided the basis and intuition for a large number of vibrational systems of all scales, ranging from the quantum to the astronomical.

We note that the standard definition of frequency (Eq. 1) is inherently ambiguous with respect to the time interval that it covers, as well as to whether it relates to a constant and time-independent frequency or to an average value (Fig. 2 A–C). These issues will become important further in the analysis.

[13] According to *Euler's formula*, $e^{ix} = \cos x + i\sin x$. The argument of the exponent in Eq. 8 appears with either a plus or a minus sign in different texts.

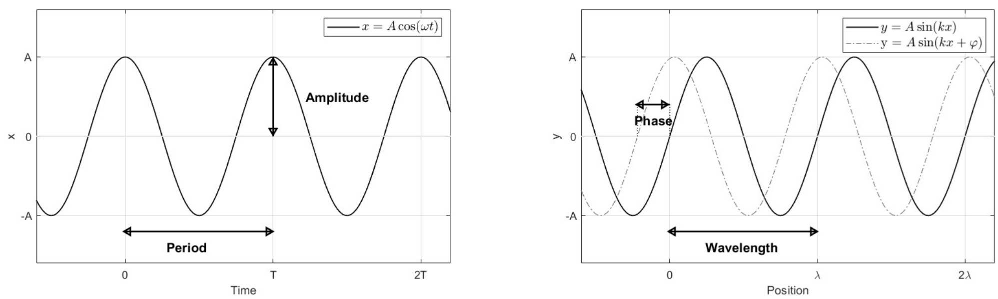

Figure 4: The basic parameters of oscillatory motion. **Left**: An example for a solution of the form of Eq. 6, with $x = A\cos(\omega t)$, illustrating the period $T = 1/f = 2\pi/\omega$ and the amplitude $x = A$. **Right**: A similar simple wave motion solution at $t = 0$, $y = A\sin(kx - \varphi)$, illustrating the wavelength $\lambda = k/2\pi$ (analogous to the period in spatial coordinates), and the initial phase φ, which in this case is negative.

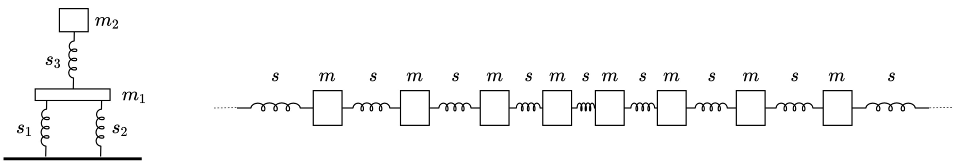

Figure 5: **Left**: An example of a mass–spring system with three springs and two masses. **Right**: Approximation of one-dimensional wave motion using identical mass–spring building-block model.

3.1.2 Coupled simple harmonic oscillators

Simple harmonic oscillators may be combined into more elaborate systems that contain multiple masses and springs, capacitors and inductors, etc. (Fig. 5). Parts of the system are coupled to the others, in a way that can be studied using systems of linear differential equations of the form of Eq. 5 (e.g., Morse and Ingard, 1968; Goldstein et al., 2014). The resultant oscillatory system can then be characterized by a set of resonances that contains as many frequencies as are degrees of freedom in the system—its *modes of vibration*. These frequencies are functions of the individual free-oscillating frequencies of the single simple harmonic oscillators. The total oscillation can be described as a superposition of oscillations at the component frequencies, which also depends on the particular initial conditions and parametric values of its building blocks. The set of all frequencies along with their amplitudes and phases makes the *frequency spectrum*, or simply the *spectrum*, of the system, which, in this case, is discrete.

3.1.3 Simple wave motion

In the limit of infinitely many identical coupled oscillators (Figs. 5, right and 6, left), it is possible to arrive at a description of continuous wave motion (e.g., of a string)—an oscillation that is periodic in both space and time and has similar mathematical solutions to the harmonic oscillator (Morse and Ingard, 1968, pp. 80–91). The simplest wave equation—the string equation—is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{9}$$

where the transverse motion in one spatial dimension (y), the string amplitude, depends on both (the perpendicular) spatial dimension x and on the time dimension t (Fig. 6, right), with the wave speed c. The wave frequency is related to its *wavelength* λ via

$$f = \frac{c}{\lambda} \tag{10}$$

Hence, for a known c, the frequency contains the same information as the wavelength. The general solution for the partial differential equation 9 is of the form

$$y(x,t) = g(x - ct) + h(x + ct) \tag{11}$$

with two functions in superposition, describing a forward propagating wave $g(x - ct)$ and backward propagating wave $h(x + ct)$. The exact solution is then determined by the boundary conditions of

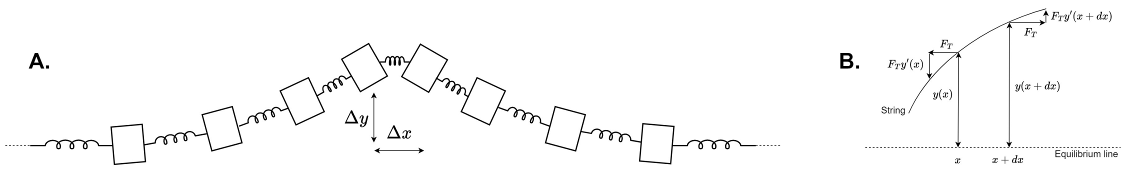

Figure 6: **A.** Approximation of a string using identical mass–spring building blocks. **B.** A geometric construction for the derivation of the string equation, using tension forces (with horizontal tension F_T in equilibrium and vertical tension producing the oscillation) in two dimensions (drawn after Morse and Ingard, 1968, p. 98).

the string. For example, when it is of infinite extent, the solution takes the form:

$$y(x,t) = A\cos\left[\frac{\omega}{c}(x-ct)-\phi\right] \tag{12}$$

with A and ϕ being the amplitude and phase of the forward-propagating wave (there is no backward propagating wave), which depend on the initial conditions. As before, the solution can also be expressed using

$$y(x,t) = \mathrm{Re}\left[Be^{\frac{i\omega}{c}(x-ct)}\right] \tag{13}$$

where the complex amplitude B now incorporates also the initial phase.

The string equation also describes other types of one-dimensional wave motion in arbitrary media where the *spatial frequency* k (also called the *wavenumber*) is defined as

$$k = \frac{2\pi}{\lambda} \tag{14}$$

which hints that the wavelength is analogous to the period, only in spatial dimensions (k is analogous to $\omega = 2\pi/T$; see Fig. 4, right). The spatial and temporal frequencies are both related through the velocity c

$$c = \frac{\omega}{k} \tag{15}$$

In all other media except for vacuum (in the case of electromagnetic waves), the propagation speed in the medium depends on the frequency of the wave, which is referred to as *dispersion* (Brillouin, 1960). This dependence can be generically expressed through either one of these two equivalent *dispersion relations*:

$$k = k(\omega) \tag{16}$$

or

$$\omega = \omega(k) \tag{17}$$

Although the effect of dispersion can be neglected in the majority of practical cases—in which case the *dispersionless* expression 15 holds—it can become important in the propagation of multiple frequencies through the medium over substantial distances. As may be seen in Fig. 2 and will be made clear in §3.4 and subsequent sections, the combination of different frequencies over a broad frequency range, at exactly the right phases, is what gives rise to complex wave geometry. These *wave groups* are sensitive to the effect of dispersion, which tends to deform their shape over distance, in proportion to the magnitude of the dispersion at the different frequencies, the distance traversed, and the frequency range of the associated group. The group itself moves at *group velocity* (Hamilton, 1839)

$$v_g = \frac{d\omega}{dk} \tag{18}$$

that is generally distinct from the *phase velocity*, $v_p = c$, of the individual frequency components that make it (Eq. 15). The concept of wave groups apply very generally to all types of waves that

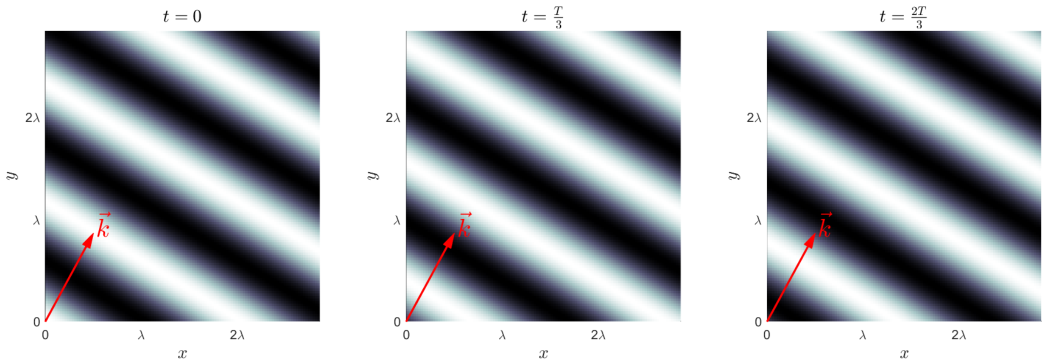

Figure 7: An example of a plane wave in two dimensions at three different time points. The temporal frequency is related to the wavenumber (the spatial frequency) through $c = \omega/k$. The direction of the propagation is determined by $\vec{k}$, which is perpendicular to the wave front (Eq. 23 with $k_z = 0$) (its magnitude is not to scale in the figure). The magnitude in grayscale corresponds to the (real part of the) field function, $\psi = Ae^{i(\omega t - \vec{k}\cdot\vec{r})}$, in arbitrary units, whose identity depends on context: displacement, velocity, pressure, electric field, etc.

are not made of only a single frequency and can appear in different names according to context—wave packets, pulses, bursts, impulses, envelopes, and modulations—all of which move at $v_g \neq v_p$ in dispersive media.

Using the solution form in Eq. 13 for the string / one-dimensional wave equation (9) enables separation-of-variables type of solution to the spatial and time-dependent terms, so that $y(x,t) = y(x)e^{i\omega t}$. This solution is said to have *harmonic time-dependence*, and in this case the remaining spatial equation simplifies to the one-dimensional homogenous *Helmholtz equation* (von Helmholtz, 1860) that is encountered in numerous places in physics

$$\frac{\partial^2 y(x)}{\partial x^2} + k^2 y(x) = 0 \tag{19}$$

Therefore, Eq. 9 can be brought to the same form as the simple harmonic oscillator (Eq. 5)—an ordinary instead of a partial differential equation. Correspondingly, the period of an ideal string of length l takes the same algebraic form as in the harmonic oscillator with

$$T = 2l\sqrt{\frac{\mu}{F_T}} \tag{20}$$

with F_T being the tension force in the string and μ its mass per unit length.

In systems of two and three spatial dimensions, the wave equation is more complex, as it admits oscillations that are distributed in all dimensions. The three-dimensional wave equation is

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{21}$$

with the *Laplace operator* $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coordinates, for example. This linear homogenous scalar wave equation is solved for some field function $\psi(x,y,z,t)$, whose exact identity depends on the medium and the type of wave. In the case of sound waves it is pressure, velocity, or density. In the case of electromagnetic radiation, it is electric field or magnetic field. And it is the displacement in elastic waves. The general solution here is of the form $\psi(x,y,z,t) = \psi_1(\alpha x + \beta y + \gamma z - ct) + \psi_2(\alpha x + \beta y + \gamma z + ct)$, with the constraint of $\gamma = \sqrt{1 - \alpha^2 - \beta^2}$. For the basic cases (no loss, no sources, everything is linear), the solutions retain the same form as in Eq. 13,

$$\psi(x,y,z,t) = \psi_1 e^{i(\omega t - \vec{k}\cdot\vec{r})} + \psi_2 e^{i(\omega t + \vec{k}\cdot\vec{r})} \tag{22}$$

for a field defined by the vector $\vec{r} = (x,y,z)$ (see Fig. 7 for a two-dimensional example). In the most general case, $\vec{k}$ is the propagation or *wave vector*, whose magnitude is the wavenumber $|k| = 2\pi/\lambda$,

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} = \frac{2\pi}{\lambda}(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}) \tag{23}$$

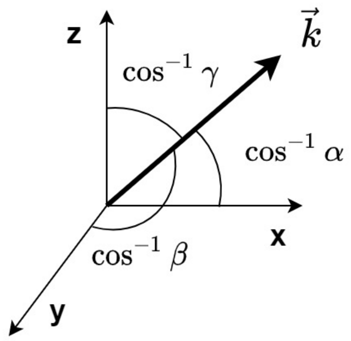

Figure 8: The wave propagation vector $\vec{k}$, whose magnitude is the wavenumber and its direction can be expressed by three angles, obtained from the inverse of the direction cosines, which together produce up to three spatial frequencies (Eq. 23; plotted after Figure 3.9, Goodman, 2017).

where α, β, and γ are the direction cosines related through the condition on γ, which satisfies the general three-dimensional solution (Fig. 8; e.g., Goodman, 2017). The directional components are then three spatial frequencies, k_x, k_y, and k_z which may be independent of one another.

The oscillations are considered *free* if there is no external force on the system. When the system is finite—as in the classical case of a string, bar, or membrane—the solutions to the wave equations are often given as a superposition of series of allowed (resonance, *natural, normal*, or *eigen-*) frequencies ω_n, each of which is differently distributed in space (see §3.4.1).

3.1.4 The simple frequency

All of the idealized systems above admit discrete parametric frequencies, which are theoretically knowable at an arbitrary level of precision. Because these oscillations do not lose any energy, they describe a state of equilibrium, where no internal or external forces disrupt the motion periodicity, and thereby affect its frequency content.

The basic definition of frequency as a reciprocal of the period (Eq. 1), which itself depends on other parameters, goes back to the seminal works on the string by Mersenne (1636) and the pendulum by Galileo (1638 / 1914) and is universally found in introductory physics textbooks. According to this definition, the period and frequency are equivalently informative, so measuring only one of the two is sufficient. In theoretical calculations, to avoid circularity—where the period is determined by the frequency and the frequency is determined by the period—it may be necessary to resort to a specific parametric estimation (as in Eqs. 2, 3, 4, and 20).

The temporal regularity of such simple physical systems and others makes them the basis for time measurements. For example, the sundial, pendulum, spring, quartz, and atomic clocks are all based on periodic systems that are highly stable, within some degree of precision[14]. To turn the periodic system into a clock, it is necessary to add a counter, which provides the information about the time elapsed according to the number of periods counted from an initial reference moment. The degree of precision of the clock increases as the oscillation period decreases, but ultimately, evaluating the precision of a finer-unit clock requires other clocks or carefully dated events with known precision that can serve as a reference.

These considerations foreshadow an understanding that both time and frequency can be ultimately used to describe the same physics and may be seen as equivalent.

Although the wave description of physical systems tends to be the most accurate one, it will be easier to focus in the following on one-dimensional oscillators, whose solutions share many similarities to wave motion, as was seen through the similarity between the one-dimensional string and the simple harmonic oscillator equations. Wave propagation and dynamics have been formulated in a large number of partial differential equations at a much higher degree of complexity than is presented here (e.g., Whitham, 1999). It is, however, possible to adopt **an observer's point of view** that is

[14] In contrast, systems like the hourglass and water clock work on an aperiodic principle, for which a decay event constitutes the time duration unit. Another time measurement is based on the decay of radioactive isotopes. This is a probabilistic event of an ensemble of particles, whose half-life constitutes the time duration unit, expressed as the time constant of an exponential decay.

more readily captured by the signal-processing approach, which quantifies and describes arbitrary time signals received at the point of detection, that is largely agnostic to the particulars of the oscillatory system. Signal processing techniques often deal with time signals at fixed positions, which means that the contributions of spatial variations through the $\vec{k}\cdot\vec{r}$ term turn into a constant (time-invariant) phase that can be incorporated into the initial conditions. Similarly, in spatial signal processing using a spatial array of measurement positions, time can be fixed ($\omega t = \text{const}$), which may translate to constant phase differences between the array points. In more complex systems, fewer parameters remain constant, and yet local measurements can still be subjected to signal processing analysis, which in our case entails also spectral analysis, as is discussed below.

3.2 Damped oscillations: relaxing the condition of strict periodicity

Relaxing one level of idealization, the simple harmonic oscillator model becomes much more universally applicable when losses, or *damping*, are incorporated into the oscillatory motion[15]. For example, the harmonic oscillator (with mechanical damping), the pendulum (with friction), and the RLC (resistor–inductor–capacitor) circuit (Fig. 9) can all be seen as embodiments of this linear ordinary differential equation:

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = 0 \tag{24}$$

where the addition of the term of the first derivative of x to Eq. 5 represents the damping, as long as $r > 0$, which entails dissipation of energy from the system. r and ω are determined by the various physical characteristic parameters of the systems (e.g., capacitance, inductance, mass, stiffness, resistance). The general solution here is of the form

$$x(t) = x_0 e^{-rt}\cos(\omega_d t - \varphi_0) \quad t > 0 \tag{25}$$

where the additional exponential term e^{-rt} represents the *envelope* of amplitude decay due to loss of energy at rate r from its initial magnitude x_0 at $t = 0$, and ω_d is a lower frequency than that of the simple (*lossless*) harmonic oscillator ω, given by

$$\omega_d = \omega\sqrt{1 - \left(\frac{r}{\omega}\right)^2} \tag{26}$$

As before, equivalent solution forms are available, similarly to Eqs. 7 (separate sine and cosine terms) and 8 (complex exponential). The motion described by these equations is, strictly speaking, aperiodic, as the amplitude of the motion decreases with every oscillation (Fig. 2 D). Therefore, a narrow definition of periodicity would consider the application of the notion of frequency inadequate (e.g., Morse and Ingard, 1968, p. 41), given to the mathematical definition of periodic functions

$$x(t + nT) = x(t) \qquad n = 0, \pm1, \pm2... \tag{27}$$

for all t and period $T \neq 0$. However, frequency can be readily salvaged, if it is computed only with respect to the phase of the motion, irrespective of the decaying amplitude. Then, it would be periodic, as the general solution can be expressed in a similar form to the simple harmonic oscillator, only at frequency ω_d rather than ω. Nevertheless, unlike the free oscillation, the damped oscillation is not defined at $t < 0$, at a time when energy must have been imparted for the motion that dissipates it later on at $t \geq 0$.

A rearrangement of the solution (25) allows for the definition of so-called *complex frequency*, which includes both the exponential amplitude as well as the periodic sinusoidal term (Bode, 1945, pp. 18–30)

$$s = \omega + ir \tag{28}$$

The advantage of this quantity is readily seen by plugging it in Euler's formula

$$e^{ist} = e^{-rt}\left[\cos(\omega t) + i\sin(\omega t)\right] \tag{29}$$

Now the right side of the equation can be used to form a complete solution that includes both oscillation and damping terms, such as the one in Eq. 25. Using somewhat different reasoning,

[15] Euler (1750 / 2021) originally referred to the simple harmonic oscillator without damping (§3.1) as belonging to the "*first kind*", the damped oscillator as the "*second kind*", and the driven oscillator (§3.3) as the "*third kind*".

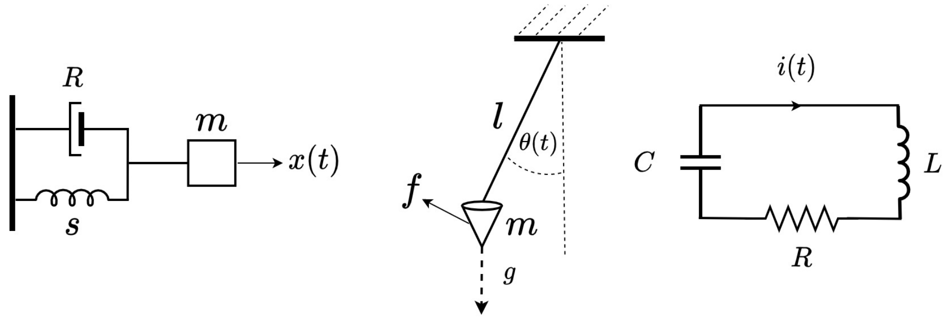

Figure 9: Three types of harmonic oscillators with loss (damping). **Left**: mass–spring–damper. **Middle**: pendulum with friction from air. **Right**: resistor–inductor–capacitor (RLC) circuit.

this solution form has become prevalent in network analysis and control theory using the Laplace transform (see §3.4.4), where cases in which $r < 0$ are of interest too, designating parametric regions of system instability, as may be the case in some situations implied in the next section. Despite its engineering usefulness, the complex frequency is largely a mathematical convenience that binds together the real frequency parameter and the real decay constant as one complex number that greatly simplifies the analysis of linear networks, such as electronic amplifiers and active filters.

3.3 Driven oscillations: the beginning of frequency time-dependence

As is apparent from the damped harmonic oscillator response, any oscillation will eventually cease, or become negligible and immeasurable, after enough time has elapsed. Therefore, in order to set such a system in motion in the first place, it is necessary to force it out of its resting state[16]. Thus, the next level of complexity that is added is the effect of an external force, which in one dimension can be described using the inhomogeneous ordinary differential equation (Morse and Ingard, 1968, pp. 43–60)

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = F(t) \tag{30}$$

The general solution of this equation consists of two terms: the solution to the homogenous equation that is independent of $F(t)$ as was given in Eqs. 24 and 25, and a particular solution that depends on the force $F(t)$. Unlike the differential equations reviewed in §3.1 and §3.2, Eq. 30 explicitly depends on time through $F(t)$ (it is a *nonautonomous* equation), so the particular solution is dependent on the absolute time point and not only on the elapsed time. Therefore, the oscillator response can be seen as a superposition of the *free oscillation*—a *transient* component that eventually dies out—and a *forced response* that receives energy for its motion from the external force. The specific force $F(t)$ that drives the oscillator, and hence the particular solution to Eq. 30, is best classified using the force's own frequency content. In general, when the force contains discrete frequencies, they also appear at the output of the oscillator, only with modified amplitude and phase compared to how they appear in the force. For example, a force that contains several frequencies that are close in value can give rise to an amplitude modulation type of oscillation (Fig. 2 E). If the force is impulsive—describing a pulse, burst, explosion, or another unspecified brief disturbance—then it starts the oscillator and gives rise to the transient response as in the damped oscillator case (Eq. 25). The force can also be random, which means that it does not contain any discernible frequencies, but only a certain *bandwidth* in the spectrum[17], or "white noise", so it has to be treated using statistical methods (see § 3.5.3). In this case, the response mirrors the undamped (free) oscillations of the

[16] Note that the words *forced* and *driven* are used more or less interchangeably in mechanics. In the context of communication and signal processing, it has become more common to refer to *modulation*, which entails that the force dynamically varies the amplitude, phase, or frequency of the oscillator through one of its parameters, often at a slow rate relative to the oscillator frequency.

[17] It is common to distinguish between *narrowband* and *broadband signals*, which occupy either narrow or broad frequency ranges, or *bandwidth*, in the spectrum, respectively. The definition is not always clear cut and it depends on the application, and in many cases, on the filters and the quantitative methods that are involved in analysis. Roughly speaking, narrowband signals are similar to pure sinusoidal oscillations, generally allowing for some fluctuations around the center frequency. Broadband signals can contain multiple, well-separated frequencies—sometimes over an infinite bandwidth—that can be represented mathematically as sums. Broadband signals can also constitute continuous range(s) of frequencies that are better treated statistically as ensembles of frequencies rather than as a sum of discrete frequencies. These distinctions will become more meaningful in the subsequent sections.

system as in Eq. 6. Forces are often described using the piecewise *Heaviside step function*,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{31}$$

that can be used in modeling the discontinuous switching-on moment by multiplying the force function, $F(t)u(t)$. Taking advantage of the presumed linearity of the system, the general classification into basic force types is particularly useful, since many (if not all) arbitrary forces can be represented as a superposition of these basic types.

Evidently, unlike the free and damped frequency definitions, forced motion may produce frequencies that are observable only from a certain time point as dictated by the external force. This understanding clashes with the requirement for constant (time-independent) frequencies that was entailed by the simple and damped harmonic oscillators. And yet, it is nevertheless possible to retain the definition in which frequencies are time independent and strictly parametric, using a superposition of infinitely long constant frequencies through the Fourier analysis, as is explained below.

3.4 Fourier analysis: Frequencies that never die out

Fourier analysis is an indispensable set of mathematical tools in the study of all oscillatory phenomena. It originally began from the Fourier series for the study of the heat equation in bounded systems, where it was applied to the string equation as well (Fourier, 1822 / 2009). In the limit of an unbounded system, the series can be generalized into an integral—the Fourier transform—which has been foundational for the analysis of continuous phenomena[18]. Out of all the integral transforms that are routinely used in harmonic analysis, Fourier analysis rules supreme due to its relative simplicity, comprehensive theory, and its recurrent emergence in the solutions of different physical problems. Less common integral transforms all generally rely on the same template, in which the product of an arbitrary function $g(t)$ and a kernel function $K(t, \omega)$ is integrated over the entire domain to yield the inverse-domain representation of the function, $G(\omega)$. Thus, the majority of the points drawn in the analysis below hold for related transforms, without loss of generality.

Fourier analysis appears in at least three distinct but complementary contexts in the standard curriculum of physics and engineering. First, the Fourier transform organically appears in the derivation of several solutions in wave physics, such as the diffraction integral in optics (Duffieux, 1946 / 1983) or in quantum mechanics (Heisenberg, 1925, 1927). Second, along with the closely-related Laplace transform, it is presented as a powerful method for the solution of linear ordinary and partial differential equations of the kind that was reviewed above (Sommerfeld, 1949), which captures the essential dynamics of all oscillatory phenomena. Third, it is a critical tool in signal processing theory, which is used to analyze arbitrary time signals following measurements and synthesis. A related usage is to apply Fourier analysis to get a handle on patterns in the reciprocal domain of various periodic phenomena (e.g., the reciprocal lattice of crystals, defined by the reciprocal of the distances between the lattice points; Ewald, 1921).

We will show how the lack of attention in the transition between the Fourier series and the Fourier transform can hint at the idea that frequency and time are one and the same thing. While the understanding that time and frequency are two separate dimensions is well-ingrained in the modern study and applications of time–frequency analysis (§3.5.6), it is not nearly as obvious from the physics literature, which does not dwell on the definitional intricacies of frequency.

3.4.1 The Fourier series and local or infinite periodicity

Discrete spectra that are composed of a series of eigenfrequencies coincide well with the method of *Fourier series* expansion, which enables periodic solutions of differential equations, as naturally arise for Eq. 5 or 21 with certain given boundary conditions that are defined on a bounded geometric interval. Because of the basic wave relations between the wavelength and the frequency, any spatial periodicity also results in corresponding temporal periodicity. However, in line with our focus on

[18]The Fourier series preceded the transform in Fourier's own text and is always presented first in introductory texts, being simpler to grasp and motivate. However, the transform itself can be universally applied to aperiodic systems as well and may therefore be more general. Thus, it is arguable whether this presentation order, which is followed here too, is indeed the optimal one (Obradovic, 2024). See also discussion of realistic acoustic sources in Weisser (2021, pp. 54–75).

time signals, we shall strictly focus on the temporal series and not deal with the analogous spatial series, which is mathematically identical otherwise.

The Fourier series for a piecewise smooth periodic function $x(t)$ over the interval $[t_1, t_2]$, with the period set to $T = t_2 - t_1$ takes the form

$$x(t) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) \tag{32}$$

where the *Fourier coefficients* a_n and b_n are determined by

$$a_n = \frac{2}{T}\int_{t_1}^{t_2} x(t)\cos\left(\frac{2\pi nt}{T}\right)dt \qquad n = 1,2,3,... \tag{33}$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_2} x(t)\sin\left(\frac{2\pi nt}{T}\right)dt \qquad n = 1,2,3,... \tag{34}$$

and for $n = 0$

$$a_0 = \frac{1}{T}\int_{t_1}^{t_2} x(t)dt \tag{35}$$

The series 32 converges to the original function[19]. The periodicity gives rise to a series of frequencies $f_n = \frac{n}{T}$, corresponding to angular frequencies $\omega_n = \frac{2\pi n}{T}$. When $n = 1$ and $f = \frac{1}{T}$, it is referred to as the *fundamental frequency*, while higher frequencies with $n > 1$ are its *harmonics* (see Fig. 10 for an example). The parameter a_0 represents the mean of the function, which is a constant by definition, and is often referred to as its *DC level*, borrowing from electricity[20]. By virtue of the periodicity of all of its components, the Fourier series naturally extends to the entire time domain $(-\infty, \infty)$ and retains its periodicity in T throughout. Example of the kind of oscillations that can be modeled with Fourier series analysis are given in Figs. 2 F and 10.

Both the original function $x(t)$ and its Fourier series representation are explicit functions of time. Only that the variable t here represents only a segment of the time axis that overlaps with a period T corresponding to the time interval $[t_1, t_2]$. The period is akin to a ruler that is positioned in space to measure the length of an object. Due to the inherent periodicity of the trigonometric functions, the Fourier series is mathematically agnostic as for how much of the time axis is covered by the same ruler shifted, so it can just as well cover the entire time domain with infinitely many periods. For this to work, we can think of the time as being mapped on the unit circle and varying between 0 and T, so that $t = T\phi/2\pi$, where the phase is bounded on the interval, $0 \le \phi \le 2\pi$. Therefore, unless we count the number of periods within our extended function where $t > t_2$ and $t < t_1$, we are only able to uniquely represent a short temporal duration of $t(\mathrm{mod}\, T)$[21]. The temporal ruler is essentially a single clock unit—a single tick—that measures time using a fixed period, which can be subdivided as needed using shorter periods.

In the Fourier series there is no ambiguity in the relationship between the time variable t and the frequency. It is clear that t only serves as a parameter to locally track the phase within the period being analyzed. Using the phase wrapping property of the periodic trigonometric functions (see Footnote 21) allows the extension of the functions over the entire time domain at no extra effort. This entails a strong assumption that the period of the extended function remains unchanged over the entire domain of t. This is tantamount to the requirement that no energy will be lost and no external forces will be applied at any moment in the past or future and disrupt the system motion. However, if we subscribe to the belief that the infinite past and future may not be knowable, this assumption is hardly realistic. It means that time and frequency may only be thought of being the same in a very limited and local sense (both in time and space) just as in §3.1. Hence, under the Fourier series analysis, both frequency and period are parametric, so time and frequency convey the same information, but only in a very restricted sense.

[19] More precisely, the Fourier series converges to the average value between the limits of each value of x, so that it is equal to $(x^+ + x^-)/2$, including at the limits t_1 and t_2, if $x(t_1) \neq x(t_2)$. In general, the integral over the period must be finite ($\int_{t_1}^{t_2} x(t)dt < \infty$)—a reasonable condition for well-behaved physical functions—and the series has to converge in order for the Fourier series to exist. *Convergence almost everywhere* between the series and the function is ensured for functions that are square integrable (L^2 functions), or $\int_{-t_1}^{t_2} |x(t)|^2 dt < \infty$ (Carleson, 1966; Lacey, 2004). These intricacies are beyond the scope of this discussion, which is concerned with the evolution of the concept of frequency, in part through the wide application of the Fourier analysis. See Folland (1992, pp. 71–82) for more details about the convergence of the Fourier series.

[20] Direct current (DC) electric power sources such as batteries are idealized in standard models as a constant voltage with no frequency components, although the battery parameters too have a time course, which is relatively slow.

[21] By virtue of the *phase wrapping* property of the trigonometric functions: $\sin(\phi + 2n\pi) = \sin(\phi)$ and $\cos(\phi + 2n\pi) = \cos(\phi)$, for any integer n.

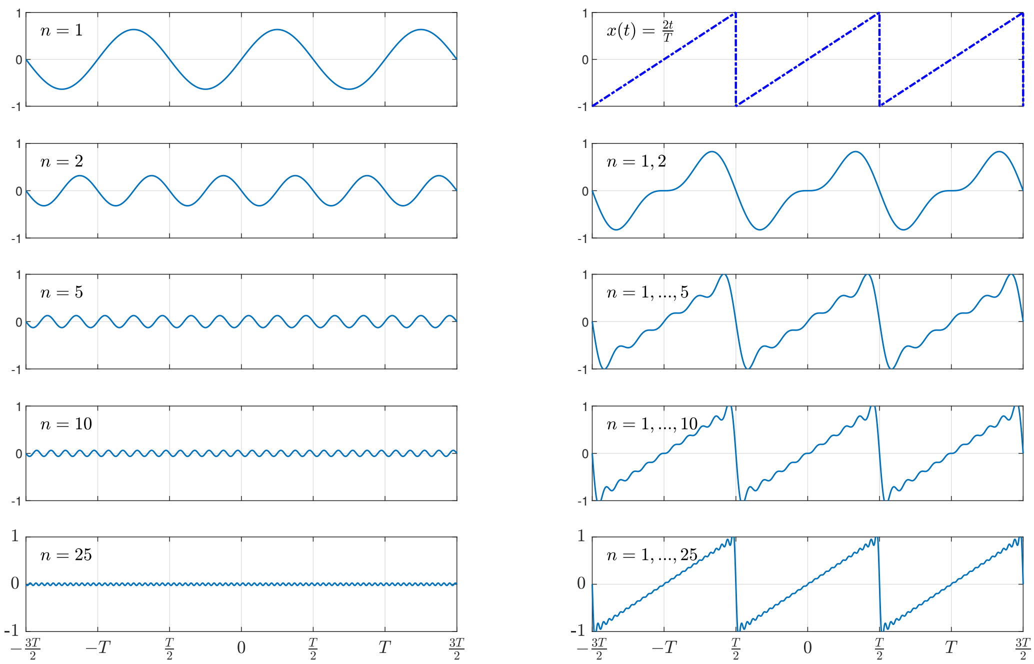

Figure 10: An example of Fourier Series of a *sawtooth wave* with a period T, $x(t) = \frac{2t}{T}$, centered on the interval $(-\frac{T}{2}, \frac{T}{2}]$ (top right) at four levels of approximations of the infinite sum, using 2, 5, 10, and 25 terms in the summation (on the right column). Each summation combines harmonics with gradually diminishing amplitudes (left column). The formula for the Fourier series is $x(t) = \frac{2}{\pi}\Sigma_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\omega t}{T}\right)$. It can be seen that as more harmonics are added to the summation, the line becomes more straight with finer oscillations, which nevertheless grow around the discontinuity at the edge of the interval, in what is referred to as the *Gibbs phenomenon* (Wilbraham, 1848; Gibbs, 1898, 1899; Hewitt and Hewitt, 1979).

3.4.2 The Fourier transform, aperiodicity, and the convenience of "zero frequency"

Things become more complicated when the Fourier series is generalized to the Fourier transform, as the functions involved cover the entire time domain rather than a finite interval in which the period is well-defined. This enables the expression of aperiodic functions as the superposition of a continuum of periodic functions with known frequencies (for example, see Fig. 2 G). To obtain the transform, it is instructive to rewrite the Fourier series in its exponential form that is equivalent to Eq. 32,

$$x(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{-\frac{2i\pi n}{T}t} \tag{36}$$

where the coefficients c_n are given by

$$c_n = \frac{1}{T}\int_{t_1}^{t_2} x(t)e^{\frac{2i\pi n}{T}t}dt \tag{37}$$

The original function can then be reconstructed by putting together the two expressions 36 and 37

$$x(t) \sim \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{t_1}^{t_2} x(\hat{t})e^{\frac{2i\pi n}{T}\hat{t}}e^{-\frac{2i\pi n}{T}t}d\hat{t} \tag{38}$$

In the limit of a very large period that covers the entire time axis, $2\pi/\omega = T \to \infty$, whereas $1/T \to \delta\omega/2\pi$, which enables the substitution of the infinite sum with a continuous integral, yielding the *Fourier integral*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\hat{t})e^{i\omega\hat{t}}e^{-i\omega t}d\hat{t}d\omega \tag{39}$$

where the small frequency interval $\delta\omega$ was replaced with the differential $d\omega$. The *Fourier transform* $\mathcal{F}$ of $x(t)$ is then defined as

$$\mathcal{F}[x(t)] = X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \tag{40}$$

with $X(\omega)$ being the *Fourier spectrum*, which is generally a continuous function. Similarly, the *inverse Fourier transform* is[22]

$$\mathcal{F}^{-1}[X(\omega)] = x(t) = \int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega \tag{41}$$

This operation is sometimes referred to as *reconstruction* or *synthesis*, whereas the Fourier transform itself (40) is *analysis* or *decomposition*[23].

In the limit of $T \to \infty$, the Fourier transform yields a continuous function of frequency, so frequency is no longer a discrete set of parameters as in all previous cases. Short of setting the Fourier integral limits at infinity to capture the signal and time in their entirety, the underlying Fourier series periodicity creeps in, and gives rise to a periodicity artifact—the appearance of phantom periodic replicas of the time signal due to insufficient coverage of the time domain. Therefore, finite approximations to the theoretical transform are defined over a single period only, which encompasses the signal duration, in consideration with its bandwidth (§3.4.3). In general, in short-time applications it is customary to limit the time interval of the signal through windowing (see §3.5.6). An example of the application of the Fourier transform on an aperiodic signal and the effect of diminishing $\delta\omega$ (and an increase of T toward infinity) and phantom periodicity is illustrated in Fig. 11.

The limit in which $T \to \infty$ results also in mapping of the time axis onto the complex unit circle. In particular, the transform includes the special value of $\omega = 0$, or *zero frequency* (DC), whose infinite period $T = \infty$ spans the entire time domain. Just like the a_0 (or c_0) coefficient in the Fourier series, it essentially refers to the mean of the time function, only that here, for mathematical convenience, it stitches the integration around zero to make frequency continuous between the negative and positive portions of the frequency axis $(-\infty, 0^-)$ and $(0^+, +\infty)$ (or, less elegantly, it connects the period axis at $T = -\infty$ and $T = +\infty$). However, zero frequency is an oxymoron—it refers to a mean value of a constant, which has nothing cyclical about it and, hence, neither periodicity nor frequency in the physical sense[24]. This category mistake is not a small detail, as is argued below[25].

3.4.3 Are frequency and time the same thing?

One of the strengths of the Fourier transform is that it represents the complete signal or wave and it is not an idealization of periodicity over a fixed interval as is the Fourier series. Therefore, it provides a complete and correct reciprocal representation of the studied phenomenon. The nuanced zero-frequency limit allows for a conceptual switch between the information afforded by the time dimension and that which is given by a continuous frequency variable. It produces a frequency-domain representation in the form of $X(\omega)$ that does not depend on time explicitly. However, the

[22] Different conventions exist of the sign of the argument in the exponent and of the $1/2\pi$ constant, which is sometimes omitted or appears in the inverse transform, or as $1/\sqrt{2\pi}$ in both the transform and its inverse.

[23] The conditions on convergence of the transform and its inverse are more involved than in the case of the Fourier series where square integrability is the condition for convergence (see Footnote 19). The most well-behaved class of functions are those in $L^1 \cap L^2$—they are both absolutely integrable (belonging to L^1, for which $\int_{-\infty}^{\infty}|f(t)|dt < \infty$) and square integrable (L^2, for which $\int_{-\infty}^{\infty}|f(t)|^2dt < \infty$). The transform can nevertheless be applied to so-called *generalized functions*, such the Dirac delta function and other distributions, which may not be as well-behaved in their convergence and integrability properties. See Folland (1992) for further details.

[24] Mathematically, every constant function $x(t) = C$ satisfies the periodicity condition (Eq. 27), since $x(t + nT) = x(t)$ for all t and integer $n = 0, \pm1, \pm2...$ However, for the case of a constant function, there is no associated oscillation that corresponds to this condition.

[25] In some applications, it is common to invoke the *analytic signal*—a complex function whose real part is identical to the measured time signal, but whose spectrum does not contain negative frequencies, which are taken to be redundant (Gabor, 1946). Even in this alternative formulation, the zero frequency is always included. The complex signal can be obtained directly from the real signal through the Hilbert transform, which itself has a singularity at $t = 0$ that requires using Cauchy's principal value in order to obtain the limit (see §3.5.7). Our zero frequency is not a singularity in the mathematical sense, but rather in the physical sense—we argue that the quantities that are referred to by T and t are categorically different and it is their merging that muddies the analysis of the frequency concept. Another argument against the literal acceptance of zero frequency can be made based on the idea of "*very small zero*", which is an infinitesimally small number that is effectively taken to be zero in physics, for convenience of analysis and modeling. It must not be confused with mathematical zero, or "*essential zero*" that is realistically nonphysical (Ellis et al., 2018).

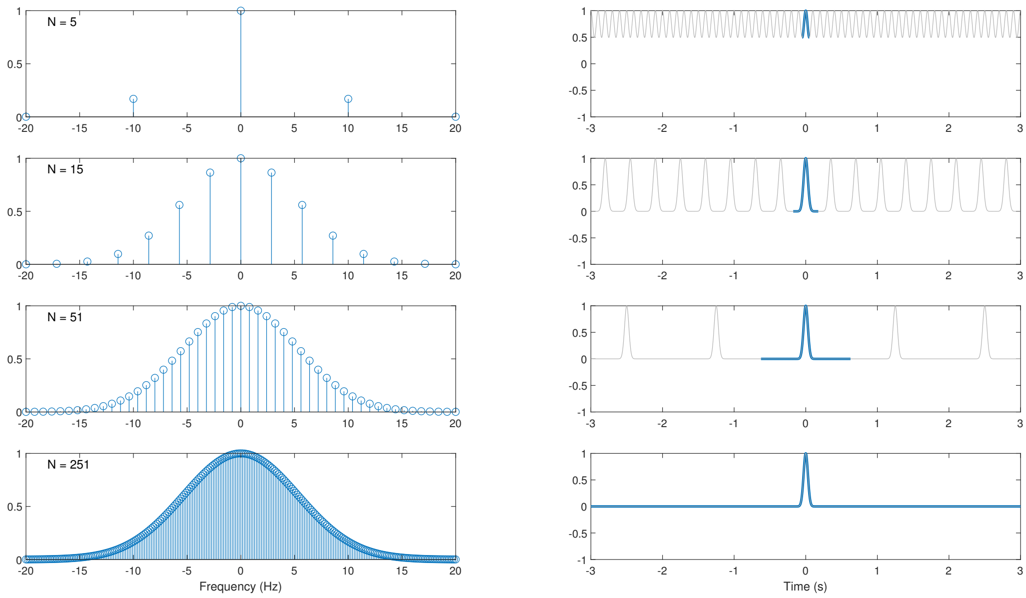

Figure 11: An example of how the approximation of the limit of the Fourier transform (39) works to cover increasing intervals from the time axis, using the reconstruction of an aperiodic signal—a narrow Gaussian (bottom right)—from the summation of an odd number Fourier series terms. The Fourier transform itself is based on the Fourier series, which is a formulation of periodic functions with period T as series of simple periodic functions. As the period T is made larger, a bigger portion of the time domain is being captured by the Fourier series, until in the limit of the Fourier transform, it overlaps with the entire time axis. In the example of the figure, the exact spectrum of the Gaussian (i.e., its Fourier transform, which is a Gaussian as well) is sampled on the frequency axis in diminishing intervals (left column). Each sample or point on the spectrum (*line spectrum*) corresponds to a single sinusoidal component in the Fourier series, which can be then summed to reproduce an approximation of the original time signal (right column). With a growing number of spectral components, $\delta\omega$ gets closer to zero, T covers a larger interval of the time axis (marked in blue at the center of the plots on the right), and the approximation gets better. The signal aperiodicity is captured in the approximation, as the increased number of components pushes the inevitable phantom periodic parts of the summation away from the Gaussian centered at zero. The phantom periods in the example are visible on the three top right time functions that are approximated by a small number of components. They disappear in the bottom right example, but they still appear out of the displayed time interval. In the Fourier transform limit of infinitely many dense components, all phantom periods disappear and the aperiodic nature of the signal is perfectly retained. Otherwise, in all other finite approximations of the reconstructed time signal, it is necessary to truncate it to a single period only, in order to avoid phantom periodicity.

Fourier transform and its inverse are characterized by near symmetry with respect to the roles of time and frequency and how they appear in the equations—only with the occasional sign change. Little but their representational nomenclature can aid us in distinguishing between the two without prior knowledge. Indeed, it is not uncommon in the harmonic analysis literature to use neutral nomenclature instead of t and ω to emphasize the variable and inverse-variable symmetry and abstract away from the specific identity of time and frequency (or space and spatial frequency; e.g., Healy et al., 2016). Now, with the transformation between $x(t)$ and $X(\omega)$ that has become habitual in analysis (for example, for the solution of differential equations), the two representations are put on equal footing—each with its own merits—and it is not unheard of to arrive at an implicit understanding that time and frequency domains convey the same information at all times, even if time is taken as the more physical quantity of the two (e.g., Blinchikoff and Zverev, 2001, pp. 2 and 27).

There are two important theorems that bolster the view that time and frequency representations are equivalent, both in terms of their total energy content and the information they carry. The first one is *Plancharel theorem*, which states that the time signal and its spectrum contain equal energy:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = 2\pi \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \tag{42}$$

This equality also holds for the more limited case of Fourier series, where the sum of the harmonic component amplitudes in the series contain the same energy as the original signal (*Parseval's theorem*) (Titchmarsh, 1948).

The second relevant theorem states that a continuous bandlimited time signal can be accurately reconstructed from a discrete time series of its amplitude values as long as it is regularly sampled at a frequency that is double or larger than the signal bandwidth. This is *Shannon's sampling theorem*—a fundamental result for all digital signal processing applications—that invokes both the Fourier transform and the Fourier series in its original proof (Shannon, 1948, 1949; see historical account in Lüke, 1999). While real signals have an infinite bandwidth, for all intents and purposes the sampling theorem provides a perfect prescription on how to discretely (*digitally*) capture continuous (*analog*) signals and release them back also as analog signals with little-to-no measurable distortion with respect to the original.

Even with these powerful theorems at hand, their usefulness is limited to the present degree of knowledge of the signal. With full knowledge of the spectrum, we also get access to perfect predictability of the time signal, and hence completely determined future, whose dynamics is expressed as the superposition of a continuum of sinusoids with constant periods[26]. The time signal and its spectrum are fully accounted for as long as all inputs to the system (external forces) or losses (dissipation of energy) are taken into account at the input to the Fourier analysis. If the energy is not conserved, then the problem formulation must be corrected so to include all the changes, in order for Plancharel theorem (Eq. 42) to still hold. Otherwise, there is nothing in the unmodified spectrum or time signal that can predict the future with unknown external effects on the system.

This reasoning entails that the idea that time and frequency are the same thing may only be entertained in the case of perfect knowledge of the time signal and its evolution, or alternatively, of the spectrum at infinite bandwidth (see Fig. 11 for a graphical illustration of this idea). No such conflation between time and frequency—really between time and periodicity—would have been possible in the first place, if it were not for the inclusion of zero frequency and the complete mapping of the time and frequency axes in the Fourier integral.

3.4.4 The Laplace transform and absolute time

A somewhat clearer relationship between frequency and time is implied by the Laplace transform (first introduced by Laplace, 1814b, Book 1, Part 2, Chapter 2), which is defined only for the positive time domain $[0, \infty)$. It is in widespread use in the analysis of linear time-invariant dynamical systems that have well-defined input and output characteristics, whose *deterministic* response can be computed once they begin oscillating at time zero (as in §3.2 and §3.3). The Laplace transform of $x(t)$ is defined as

$$\mathcal{L}\left[x(t)\right](s) = X(s) = \int_0^\infty x(t)e^{-st}dt \tag{43}$$

for complex frequency $s = i\omega + r$ as a parameter (Eq. 28). Convergence of the integral requires the existence of the integration limit $|\int_0^\tau x(t)dt| < \infty$ when $\tau \to \infty$, and depends on the choice of the real part r, which entails a specific region of convergence. Setting $s = i\omega$, the resultant Laplace transform provides the characteristic response to given inputs with known frequency content, which is generally shaped by the system itself. A general inverse Laplace transform $\mathcal{L}^{-1}$ exists as well, and it can be shown that if for a given function $x(t)$ the inverse transform $\mathcal{L}^{-1}[X(s)](t)$ exists, then the function $x(t)$ is uniquely determined by the Laplace transform itself, up to local discontinuities in the function.

The Laplace transform may be understood as a special case of the Fourier transform of a signal $x(t)$, which begins at $t = 0$ by virtue of the Heaviside function $u(t)$ (Eq. 31), and has a decaying amplitude that goes as e^{-rt}

$$\mathcal{L}\left[x(t)\right](s) = \int_0^\infty x(t)e^{-st}dt = \int_{-\infty}^\infty x(t)u(t)e^{-rt}e^{-i\omega t}dt = \mathcal{F}\left[x(t)e^{-rt}u(t)\right] \tag{44}$$

where the $1/2\pi$ was omitted. Equivalently, the Fourier transform may be understood as a special case of the (double-sided) Laplace transform when $r = 0$.

In the Laplace transform the frequency is still defined in the same way as it was in the Fourier analysis—with component frequencies that never cease and only cancel each other out. Phenomenologically, the relationship with time in the Laplace transform is always deterministic, because any

[26] This discussion anticipates the concept of determinism, which is invoked throughout the latter part of this work in the Laplacian sense, that is quoted and discussed more in depth in Appendix §A.

pure oscillation has an absolute starting point in time and will eventually die off, as long as the total damping term is positive (the progression of the exponential is decaying and it takes unique values in time). Thus, although ω can take any real value on $(-\infty, +\infty)$, as long as the transform converges, the inclusion of zero frequency does not seem to pose any serious conceptual problems in this case, because the notion of an infinite period is not instrumental in damped systems. Arguably, the spectral information garnered by the inverse Laplace transform is not going to be easily conflated with all of time, because its claim is anyway limited to the dynamic response of a system embedded in time. Invoking the time ruler metaphor again, a damped response may provide a yardstick measure of duration, like an hourglass, where periodicity is not as useful a concept for the time elapsed as it is with oscillator-based clocks.

3.4.5 The compact support paradox and the cost of complete determinism

Full knowledge, or determinism, of the signal and its Fourier transform encapsulates a deep discord with the observed reality. A well-known property of the Fourier transform is that a signal can be finite only in one domain (technically referred to as having *bounded* or *compact support*[27]). In other words, the Fourier transform of a signal with finite duration has an infinite bandwidth, whereas a signal with a finite bandwidth has an infinite duration. Slepian (1976, 1983) commented on this deeply unsatisfactory discrepancy between the mathematics and the reality in which the signals of interest in engineering are finite both in time and in frequency. He attempted to resolve this paradox by making a distinction between the abstract nature of the mathematical constructs that are used in signal analysis and Reality itself. He further suggested that it is easy to conflate the observed reality and the mathematics, but they are not the same thing. The very notion of frequency, to him, is a construct of convenience and utility that need not have any meaning for the real signal[28]. He finally went on to identify signals whose energy is effectively concentrated in finite intervals both in the time and in the frequency domains and have negligible residual energy outside of these intervals. Adhering to these signals is what enables sampling, as was prescribed by Shannon's sampling theorem, which works well in near real-time and effectively turns this uncomfortable paradox moot.

It should be underlined that there is nothing at fault with the Fourier transform itself that ushers the compact support paradox. It may describe Reality perfectly well, only at a level we have no access to: inability to garner perfect knowledge about signals in the remote past and future, and inability to measure infinitesimally small amplitudes at arbitrary frequencies, as are predicted to exist by the infinite support of the Fourier calculus.

3.4.6 The uncertainty principle

Signal determinism in Fourier transform analysis appears to be served a final blow in the form of *the uncertainty principle*, which becomes a thorny issue exactly for the signals that are not as compactly bounded as those highlighted by Slepian (1976, 1983). The uncertainty principle first appeared in quantum mechanics, where it was shown that it is impossible to simultaneously measure the position and momentum of a particle, or alternatively, to simultaneously measure its energy and time (Heisenberg, 1927)[29]. However, the uncertainty principle is a more general property of any pair of functions that are the Fourier transform of each other (as are the quantum position and momentum and the energy and time operators), as was proven by Gabor (1946) for any time signal in an analogous way to quantum mechanics[30]. In this version, the product of the standard deviations

[27] The *support* of a real function or signal is the interval along its domain (its variable) that fully contains the function's non-zero extent, beyond which it is mapped to zero.

[28] From Slepian (1976, p. 293): "*...the words 'bandlimited,' 'start,' 'stop,' and even 'frequency' describe secondary constructs from Facet B of our field. They are abstractions we have introduced into our paper and pencil game for our convenience in working with the model. They require precise specification of the signals in the model at times in the infinitely remote past and in the infinitely distant future. These notions have no meaningful counterpart in Facet A. We are no more able to determine by measurements whether a 'real signal' was always 'zero' before noon today than we are able to determine its continuity with time.*" He referred to the observed reality as "*Facet A*" and to the various analytical tools that are employed to describe and manipulate it as "*Facet B*".

[29] It is arguable whether Heisenberg (1927) actually proved the uncertainty principle in his seminal paper, where it appeared in a limited form as $\Delta p \Delta q \sim \hbar$ (Marburger, 2008; Ozawa, 2015). In his lecture series later, Heisenberg referred to a rigorous proof by Kennard (1927) that came shortly after, which does not generalize to arbitrary wave functions and is therefore flawed (Marburger, 2008). It was followed by a rigorous proof by Weyl (1928 / 1950, pp. 77 and 393–394), credited to Wolfgang Pauli, based on the Schwartz inequality. Other proofs for the uncertainty principle exist, beginning with Robertson (1929).

[30] Particular instances of the uncertainty principle had been known even before Heisenberg. One such relation from electrical signal analysis was derived by Küpfmüller (1924), who calculated the optimal bandwidth of a bandpass filter that can retain the form of rectangular Morse code pulses of certain duration. He recognized that the inverse proportion between the bandwidth and

of the time and frequency of any arbitrary signal—based on its duration and its Fourier spectral bandwidth—has a minimum,

$$\Delta t \Delta f \geq \frac{1}{4\pi} \qquad \Delta t \Delta \omega \geq \frac{1}{2} \tag{45}$$

with equality of the two equivalent forms achieved only in the case of Gaussian-shaped signals, whose corresponding Fourier spectra are Gaussian as well. In practice, the uncertainty principle constrains the precision in which the frequency content of very short time signals can be determined (see rectangular pulse examples on Fig. 22, left column). Hence, it is also referred to as the *time–bandwidth product theorem* in signal analysis. Other transforms in harmonic analysis are all constrained by similar uncertainty bounds (Folland and Sitaram, 1997; Tao and Zhao, 2016).

While the uncertainty principle in quantum mechanics has been discussed, interpreted, and contested in innumerable texts of physics and philosophy, its signal-analytic counterpart has been accepted rather matter-of-factly as an inevitable constraint to be reckoned with in applied time–frequency analysis (e.g., Cohen, 1995; Debnath, 2002). According to Cohen (1995, p. 45), the signal-analytical uncertainty principle is a misnomer that merely spells out the reciprocal relations between the time signal duration and its bandwidth, similarly to that mentioned in §3.4.5.

3.4.7 Perceptual discrepancy between time-invariant spectra and time-varying signals

As was implied above, equivalence between the time-domain and frequency-domain representations of the system dynamics is retained if all inputs and outputs of the system are accounted for. Therefore, it has been a common practice, especially in physics, to a priori specify the types of forces that drive the system in order to have no ambiguity as for how the system behaves (the simplest cases were discussed in §3.2 and §3.3; see §3.5.1). It also entails the existence of a deterministic, time-independent Fourier spectrum of the corresponding time signal associated with all outputs of the system (§A.1).

In many engineering applications, however, it is necessary to deal with time variations that are not well captured by the time-independent spectrum. The most instructive examples of dynamic signals are different forms of *frequency modulation* (FM). Although it was originally introduced as a technique for radio communication, its relevance has been shown for naturally occurring signals such as human voice and animal vocalizations (e.g., Klug and Grothe, 2010), as Doppler shift as a result of a moving radiating source (e.g. Middleton, 1977), and the general propagation of broadband pulses in higher-order dispersive media (Weisser, 2021). For example, sinusoidal frequency modulation has a time-dependent phase term

$$x(t) = \cos\left[\omega_c t + m \sin(\omega_m t)\right] \tag{46}$$

where ω_c is referred to as the *carrier frequency*, ω_m is the *modulation frequency*, and m is the *modulation index* (Van Der Pol, 1930). The time-dependent phase term implies that the signal has **frequency that varies in time**. When it is implemented as a stimulus for certain sensory modalities, it may also be perceived as such. For example, at audible frequencies with a very slow modulation frequency and large modulation index, a signal of the form of Eq. 46 sounds like a siren.

Despite its time-varying perception, the sinusoidal FM signal can nevertheless be formulated using Fourier series, as an infinite series of sinusoids, whose amplitudes scale as Bessel function of the first kind $J_n(m)$ (Carson and Fry, 1937)[31]

$$x(t) = \sum_{n=-\infty}^{\infty} J_n(m) \cos(\omega_c t + n\omega_m t) \tag{47}$$

While mathematically exact and analytically important, conceptually this is an unsatisfying result, as it does not convey the spectral changes that are perceived in real-time (e.g., Blinchikoff and Zverev, 2001, pp. 383–395). The same goes for *linear FM* (*up-chirp* or *down-chirp*), which has a

duration ought to be constrained by an unspecified universal constant that is independent of the choice of pulse shape and criterion for shape retention. In another anecdotal account, Norbert Wiener recounted a talk he gave in Göttingen in 1925, in which he discussed the uncertainty principle in harmonic analysis (Wiener, 1956, pp. 105–108). He hinted that both Max Born and his student Werner Heisenberg may have attended the talk and could have been influenced by his ideas.

[31] In the transition between Fourier series and Fourier transform, every term in the series is transformed into a delta-function pair, representing an infinitesimally narrow frequency line spectrum (i.e., $x(t) = \cos(\omega_0 t) \xrightarrow{\mathcal{F}} X(\omega) = \frac{1}{4\pi}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$ and $x(t) = \sin(\omega_0 t) \xrightarrow{\mathcal{F}} X(\omega) = \frac{1}{4i\pi}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$). While not strictly valid in the classical usage of the transform that requires finite signals with finite energy content, it nevertheless retains the physical intuition and is routinely used in practical applications. See §A.2 for further discussion.

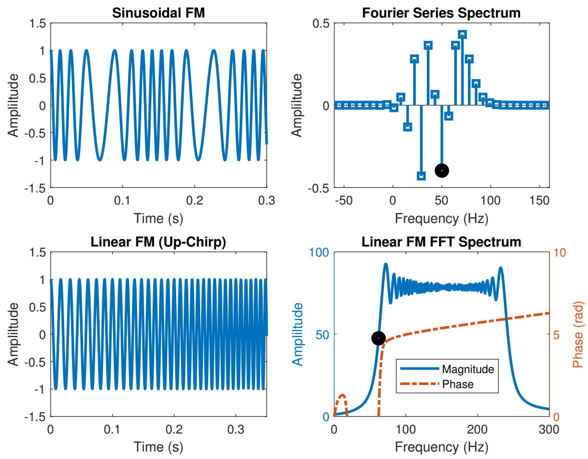

Figure 12: Two common frequency modulation (FM) time signals and their respective spectra. On the top left is a sinusoidial FM signal $x(t) = \cos[2\pi \cdot 50t + 4\sin(2\pi \cdot 7t)]$ with its Fourier series components according to Eq. 47 are displayed on the top right. On the bottom left is a linear FM signal, a rising (or up-) chirp, with the equation $x(t) = \cos(2\pi \cdot 60t + 200\pi t^2)$. The spectrum of the signal was computed using the fast Fourier transform (FFT) algorithm (credited to Cooley and Tukey, 1965, but with early roots traced back to Gauss; Cooley, 1987), whose magnitude is displayed (in blue) on the bottom right. Also displayed (in red) is the unwrapped phase variable. In both spectral plots, the carrier is marked with a black circle ($f_c = 50$ Hz for the sinusoidal FM and $f_c = 60$ Hz for the linear FM).

prohibitively complex Fourier transform (Klauder et al., 1960). Examples of these two FM signal types along with their respective spectra are displayed in Fig. 12.

While these two examples are still *deterministic* and can be treated in closed form that is analytically decomposable to component frequencies, they herald the complexity of *nonstationary signals*, which are encountered in most realistic contexts. These are signals that vary too rapidly or erratically to be captured by a precise spectrum, but not enough to be completely devoid of structure, i.e., local periodicities that correspond to well-defined spectral patterns. The distinction between stationary and nonstationary signals and spectra comes from the statistical approach to time series analysis (see §3.5.2–§3.5.5 below). In general, nonstationary signals are those that depend on the absolute time point at which they are sampled. Speech, music, earthquakes, moving optical objects, material textures that are momentarily felt by touch, or the electric current that runs a computer program in its circuitry, can all be thought of as instances of nonstationary signals and stimuli (see Fig. 15 for examples).

3.5 Beyond the classical Fourier transform

At this point we have identified several aspects of the ubiquitous Fourier transform that are not straightforwardly reconciled with reality:

- The convenience of zero frequency may lead to conflation between time and periodicity.
- Complete determinism is required in order to precisely calculate a spectrum.
- Mathematical signals have infinite support in time or frequency or both, unlike real-world signals that appear to be finite both in time and in frequency.
- The spectra of signals that have a distinct nonstationary character (perceived or measured) do not intuitively reflect their time-varying nature[32].

Many methods have been devised to overcome the final three limitations, primarily with the intent to be able to analyze nonstationary signals, often in real time. Some methods salvage the classical notion of frequency as a parameter, whereas others only use it in a statistical manner when the signals are stationary. It is not the intention to survey these methods in any detail, but rather identify some of the underlying assumptions and guiding philosophy and contrast them with the definition and status of frequency we are contending with.

[32]To this list we can add issues that are more technical in nature and do not threaten the very definition of frequency as the ones listed: the appearance of negative frequencies, the limitation of applying Fourier analysis to nonlinear systems, and the existence of stochastic signals that do not have a Fourier representation and can only be analyzed statistically (see §3.5.2–§3.5.5).

3.5.1 Retaining determinism by force inclusion

When the dynamics of an oscillatory system impacted by an external force is studied, the most common method that is employed in classical science and engineering problems is to include the applied force with the complete time signal model, so that no external forces are ever truly external to the analysis (Fig. 13). The most well-studied examples are the inclusion of loss (see § 3.2), discontinuity in applied force or medium, an impulse, a periodic force, and amplitude-modulated forces. Each one of these types of forces can be readily represented in closed form, which results in a particular (inhomogenous) differential equation. The forces lend themselves to Fourier analysis as well, so they inject their own spectral content into the system. Superposition allows for the generation of arbitrary forces based on these simple building blocks (see §3.3), which is often complemented by multiplication of the force by functions that further shape it and that transform to convolution in the reciprocal domain[33]. The Fourier transform is linear too, which means that time signals can be also produced piecewise and the total spectrum is obtained by superposition of the different parts.

Therefore, this is a "meta-method" of a sort that presupposes knowledge of all forces impacting the system. It ensures that over the evolution of its dynamics, the total energy is conserved and no new information is introduced into its analysis, as both system and external force(s) are isolated from other external influences. Here, frequency and time representations carry exactly the same information by the very statement of the problem. This is warranted, for example, in situations in which there is tight control over the signals that are generated by the experimenter or theoretician. However, as the external forces get more complex and variable, then the usefulness of this method degrades, as the resultant spectrum becomes less and less intuitive, as was seen in § 3.4.7. And, critically, if the control over a priori defined signals is limited, it makes the problem definition idealized, as this type of analysis is only as good as the predictability of external forces that can be guaranteed, along with the complete knowledge of the system history. This somewhat confusing problem statement is not generally discussed in physics textbooks, whereas openly acknowledging it has been the basis for all modern time–frequency analytical methods.

There is a sense of circularity in this important way of dealing with system dynamics, as it implies that time and frequency representations are equivalent by definition. Because, if we added another force, then it would have been an altogether different problem with its own time and frequency representations. As these problems are defined over the entire time axis, the system under analysis is effectively isolated (also with respect to losses, as any heat generated from energy dissipation remains in the system; Fig. 13 C). Once again, this way of thinking is only possible because of the subtle switch between periodicity and time in the Fourier integral, facilitated by the inclusion of the zero frequency.

3.5.2 Statistical signal processing I: Motivation

A statistical rather than a *deterministic* approach to signals has been widely used in spectral analysis within numerous disciplines of science and engineering (e.g., Middleton, 1996 / 1960; Jenkins and Watts, 1968; Bendat and Piersol, 2011). Probabilistically expressing time signals, or time series in the discrete case, is done by relating to records (or samples) that are drawn from a *process* that is defined as an indexed set of these samples $\{x_i(t)\}$ and can be either *deterministic* or *stochastic* (*random*). Referring to an underlying phenomenon as a process indicates that it is not completely under control between observations and the most that can be known about it is encapsulated in the statistical regularities that it exhibits and can be observed over time. The collection of samples forms an *ensemble*, which is characterized by an underlying probability distribution. As such, every feature that the process possesses may be treated as a statistical property as well, which has its own mean, mean square, variance, higher-order moments[34], etc. This approach applies to the corresponding

[33] The *convolution theorem* states that the product of two functions Fourier-transforms to a *convolution integral* in the reciprocal domain, so for arbitrary time functions $s(t)$ and $r(t)$ we have $\int_{-\infty}^{\infty} s(t)r(t)e^{-i\omega t}dt = 2\pi \int_{-\infty}^{\infty} S(\omega-\omega')R(\omega')d\omega'$ and the inverse $2\pi \int_{-\infty}^{\infty} S(\omega)R(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} s(t-t')r(t')dt'$. Despite its apparent complexity, the convolution operation and theorem facilitate the calculation of a cascade of two functions or more, each of which has a temporal / spectral response of its own. For example, in signal processing it is straightforward to understand what a low-pass filter does—it removes low frequencies from a signal (such as cutting the bass from sound). But to obtain the time signal of an arbitrary signal that goes through the filter, it is necessary to convolve it with the filter's time-domain function—its *impulse response*.

[34] A *moment* of order n of a function or distribution $p(t)$ is defined as the expected value of a power of the variable, $E(x^n) = \int_{-\infty}^{\infty} x^n p(x)dx$. So, the first moment is the *mean*, the second moment when taken around the mean is the *variance*, the third is *skewness*, and the fourth is *kurtosis*. Aside from characterizing the distribution, in some cases it is possible to uniquely obtain the distribution from its set of moments (when they exist).

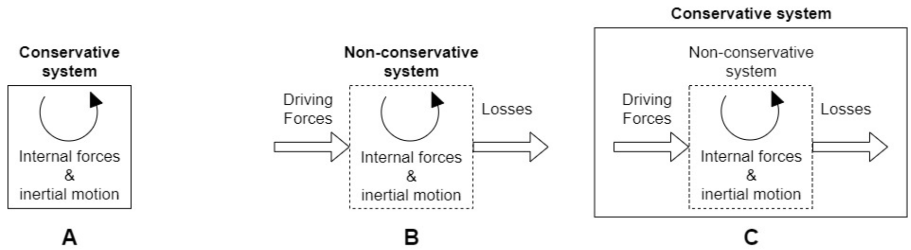

Figure 13: Three ways to model the energy balance in a dynamical system: **A**. By assuming full isolation, so there is no energy dissipation or addition and the total energy within the system boundaries is constant. This assumes that all energy sources are self-contained and any motion that is associated with the system center of mass is inertial, while internally the system is in equilibrium. This is a classical conservative system that appears stationary. **B**. By including losses and energy inputs in the models, but not as an integral part of the system, which is instead modeled as non-conservative, and hence, open. **C**. By including all losses and inputs to the system as part of yet a larger system (the environment plus the system), which is itself conservative, in the sense that the total energy is accounted for and remains within the modeled system.

frequency-domain estimates of time signals as well. The motivation to use statistical methods in modeling and analysis is extensive and only a few reasons are mentioned here:

- The data describe the aggregate behavior of numerous particles / systems / components that cannot be individually tracked and where their individual nature does not play a (critical) role at a higher level of analysis (e.g., the macroscopic behavior of many molecules, the amplitude and phase of individual wavefronts of sunlight reflected from surfaces, or the time evolution of rain that is composed of many raindrops).
- The time data appear random (like noise) or complex, perhaps without clear patterns of periodicity, and the data may not be exactly reproducible between measurement repetitions.
- The analysis applies to a sample of *deterministic* signals (or to one long signal cut into segments) that are measured over time and can be treated as though they were drawn from the same probability distribution (e.g., transmission of amplitude, frequency, and phase modulated signals with or without noise, Middleton, 1996 / 1960; the long-term average spectrum of music or speech, Voss and Clarke, 1975; Byrne et al., 1994).
- The signals are distributed over a certain bandwidth of frequencies, whose decomposition to individual components is generally impossible[35] or impractical. For example, the phase structure of ambient light, whose frequency range is too high to be measurable with standard instruments, including the retinal photoreceptors in the eye, or the characterization of ambient noise in communication equipment.
- Formation of statistics of random signal ensembles whose exact sample properties such as the specific start and end times (i.e., their relative phase and duration) are unimportant.
- Analysis of signals about which not much is known and about which there is a significant degree of uncertainty, so a generic distribution has to be employed by making certain (conservative) assumptions (e.g., stationarity, ergodicity, normality, uniformity, independence of events, memorylessness).
- Analysis of systems that are too complex to be modeled directly using *deterministic* models such as those reviewed in §3.1–§3.3.
- Analysis of systems in the presence of noise, either from the measurement instruments and methods, or from external sources, where the aggregation of more measurements leads to reduction of measurement error and hence to more precise estimation of the response of the system.
- The analysis pertains to an average system response to arbitrary inputs and outputs.

In all cases, the statistical approach represents an observer's point of view, who is interested in extracting only the most salient features from a large amount of data, or has to deal with a relative paucity of information about the measured system, or is after system response data driven by signals that are inherently stochastic, including noise.

[35] No valid Fourier transform exists for random signals that do not have a well-defined phase structure. Rather, only the signal square, proportional to its power, may have a valid distribution. See the discussion about the power spectral density in §3.5.3.

3.5.3 Statistical signal processing II: Stationarity

Arguably, the most important assumption that has been routinely invoked in the study of stochastic processes has been the one of *stationarity*. A *strict-sense stationary process* is characterized by time-shift invariance of the ensemble,

$$x_i(t) \to x_i(t+\tau) = x_i(t) \tag{48}$$

so that the mean of $x(t)$ is unchanged by the *time shift*, *lag*, or *delay* τ. In *wide-sense stationary processes*, higher-order statistics W_n are independent of the absolute time point at which they are sampled or calculated and only depend on the relative differences between times of measurement, so

$$W_n(t_1, t_2) = W_n(\tau) \tag{49}$$

with $\tau = t_2 - t_1$. Most importantly, it applies to the primary tool for identifying periodicities in any kind of process—the *autocorrelation* function—which compares the process to a delayed version of itself. For the ensemble $x(t)$ belonging to a certain stationary process, it is defined as the time average

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^*(t)x(t+\tau)dt \tag{50}$$

which is also equal to the ensemble average when the process is stationary and *ergodic*, defined so that every realization of the ensemble carries the same statistical properties of the process as a whole,

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x^*(t)x(t+\tau)dt \tag{51}$$

R_{xx} is a function of the time delay $\tau = t_2 - t_1$ between two different time points of the process. When the time delay corresponds to a periodicity of the ensemble, $R_{xx}(\tau)$ peaks. From the definition of frequency (Eq. 1), periodicity peaks should also correspond to frequencies in the reciprocal domain. Indeed, according to the *Wiener–Khintchine theorem* (Wiener, 1930; Khintchine, 1934)[36] the Fourier transform of the autocorrelation of a stationary process is the Fourier transform of its *power spectrum* (or *power spectral density*) $S_{xx}(\omega)$

$$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-i\omega\tau} d\tau \tag{52}$$

The power spectrum itself is a real function and may be defined directly from the square of the complex Fourier spectrum, as appears in Plancharel's theorem (Eq. 42)

$$S_{xx}(\omega) = |X(\omega)|^2 \tag{53}$$

This quantity has major practical value in applications where the phase response is of secondary importance. The inverse transform to the Wiener–Khintchine applies as well

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau} d\omega \tag{54}$$

Just as in the *deterministic* case (Eq. 42), integrating either the autocorrelation or the power spectral density over the entire domain, yields the total power contained in the process. Furthermore, the power bounded within a particular bandwidth is obtained by integrating $S_{xx}(\omega)$ over that bandwidth (and hence it is a density function). However, even when the bandwidth is very narrow, the mean frequency that is associated with its center may not correspond to an individual frequency of any particular sample that forms the ensemble. The reciprocation of this is that the phase of an individual sample cannot be determined from S_{xx} either, as all phase information is discarded in its calculation (Wiener, 1930, pp. 129–130). This stems from the loss of the exact timing of the individual samples that constitute the ensemble.

A critical example of these ideas is the characterization of *white noise*. This type of noise is an idealized stochastic *normal process* that describes important physical phenomena, such as thermal noise in conductors (*Johnson–Nyquist noise*), caused by the current fluctuations generated

[36] The proof of the theorem was already sketched by Einstein (1914) and appears to have been informally in use in the decades before (Gardner, 1987).

by the random movement of free electrons in the conductor (Johnson, 1928; Nyquist, 1928). For a conductor with a frequency-independent resistance R, the power spectral density function is constant at low frequencies ($f < 10^{10}$ Hz), which corresponds to a Dirac-delta-function autocorrelation (e.g., Middleton, 1996 / 1960, pp. 467–488)

$$S_{xx}(\omega) = 4k_B T_0 R \qquad R_{xx}(\tau) = 2k_B T_0 R \cdot \delta(\tau) \tag{55}$$

where $k_B = 1.38 \cdot 10^{-23}$ J/K is Boltzmann constant and T_0 is the temperature of the conductor at equilibrium.

The strict attribution of stationarity to a process is an idealization, as it implies that the underlying dynamics is in equilibrium or is at least stable with respect to forces and fluxes that act upon it from the environment. In many cases, stationarity implies conservation of energy, when the physical system is completely closed to interactions with the external environment (Landau and Lifshitz, 1980, pp. 15–16). Despite the idealization, it has been remarkably effective in modeling to assume stationarity in the analysis of various phenomena that do not appear to be in equilibrium, including in processes that might appear to violate stationarity on some time scales, but exhibit it over relevant sample durations.

3.5.4 Statistical signal processing III: Nonstationarity

Nonstationary versions of all these functions exist as well, although not in as much use as their stationary counterparts. Here the autocorrelation depends explicitly on the two absolute time points t_1 and t_2, rather than on their relative difference. It leads to interaction between frequencies in the power spectrum $S_{xx}(\omega_1, \omega_2)$ (Bendat and Piersol, 2011, pp. 442–448). For a random nonstationary process $x(t)$,

$$S_{xx}(\omega_1, \omega_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(t_1, t_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} dt_1 dt_2 \tag{56}$$

With the inverse giving the autocorrelation function

$$R_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xx}(\omega_1, \omega_2) e^{-i(\omega_1 t_1 - \omega_2 t_2)} d\omega_1 d\omega_2 \tag{57}$$

In that case, the stationary process formulation can be shown to be a special case of the much more general nonstationary process formulation, where there is no interaction between frequencies. Note that even in the analysis of nonstationary systems and processes, whose transient nature complicates any statistical analysis, it is generally attempted to decompose the samples to stationary and nonstationary components, so that at least part of the process analysis can be simplified through locally analyzing it as stationary. See also comment in §3.4.7.

3.5.5 Statistical signal processing IV: Discussion

Returning to the discussion about the dimensions of Reality, it may not serve any purpose to keep the absolute time or assign a time axis to the progression of particular samples of the stationary ensemble, but only parametrically relate to the periodicities at which they are likely to oscillate, on average[37]. Despite the loss of fine details of the process due to averaging, patterns that are characterized by relatively low frequencies can emerge from such analyses. All in all, whether because of the observed system or data, or due to the mathematical methods themselves, the statistical approach inherently incorporates a degree of uncertainty, which results in inevitable indeterminism that cannot yield the same specific timing details that a strictly *deterministic* analysis can.

The statistical approach to spectral analysis does not offer a different definition of frequency than the *deterministic* approach[38]. However, it is common to describe stochastic processes using the concept of power spectrum, where it is only meaningful to speak about the energy contained within a certain bandwidth. Thus, for a finite (discrete) frequency to exist as in fully *deterministic* cases,

[37] Note the resemblance to the parametric nature of time in autonomous *deterministic* dynamical systems, which entails time-translation invariance.

[38] It should be emphasized that the notion of *(relative) frequency* that is used in standard (frequentist) statistics (e.g., Kendall, 1945) is generally **not** the same as frequency in physics, despite occasional convergence in meaning. Relative frequency relates to the occurrence of some observation relatively to an independent parameter—not necessarily time—that reflects the underlying probability distribution in the population (e.g., the frequency of people in the population who wear capes). The one thing that is common in both usages of the term, though, is that in both cases the frequency measure is abstracted from time (at least whenever the probability distribution is itself time invariant).

it is approximated by a delta-function spectral density. The power spectrum is a key descriptor that is interwoven with the idea of periodicity, but is generally defined over an ensemble rather than over a particular signal. Nevertheless, the same relations (here, Eqs. 50, 51, 53, 54, 56, and 57) apply just as well to *deterministic* signals as they do to ensembles of signals, which offers much flexibility in analysis.

The above overview underlines three points that are pertinent to the broader discussion exploring the possible role of frequency as an independent dimension of Reality. First, the time coordinate, which is conflated with the periodicity parameter in the *deterministic* Fourier transform formalism (§3.4.3), refers strictly to the periodicity of the signal within the statistical approach and power spectrum framework. We note that a constant DC signal, which is mapped with the Fourier transform to a delta-function peak at $\omega = 0$, does not have a well-defined value of its autocorrelation function (Eq. 50). This coincides with the above discussion about the categorically ill-defined notion of zero frequency (§3.4.2 and §3.4.3). Second, there exist some systems and processes—idealized as they may be (through stationarity)—that are well-characterized using statistical processes in which time is not a dimension, but only a parameter. Third, one of the main problems that was highlighted in the *deterministic* frequency overview was that the Fourier integrals require deterministic knowledge of the remote past and future of the system (or infinite bandwidth). The statistical approach is obviously indeterministic, but the notion and assumption of stationarity implies an underlying probability distribution that is unvarying, which in idealized cases also extends to the remote past and future as the problem dictates. Nevertheless, the statistical invariance in itself, as well as the idea that a distribution can be estimated and measured, sneaks in a deterministic component to this otherwise stochastic, indeterministic perspective.

3.5.6 Time windowing

By far, the most important analytical step in curtailing the infinitude associated with the fully deterministic and time-invariant Fourier transform is the application of *windowing*, first introduced by Lord Rayleigh (1912). A *time window* is a real function that limits the duration of the time signal, so that it is forced to zero outside of a well-defined time interval. It weights the contributions of the signal at different times in the non-zero portion and completely suppresses any remote past and future contributions. Typical examples among dozens of available windows are the rectangular window, the triangular window, and the cosine square window (i.e., half a period of cosine square function). The windowed signal itself has a modified spectrum that is the convolution of the full spectrum with the Fourier transform of the window function (see examples in Fig. 14)[39].

The most typical application of time windowing is signal analysis in *time frames*—short segments of the complete signal—each of which has its own time-localized spectrum, which may not necessarily be time-varying in itself (Welch, 1967). This procedure produces a **two-dimensional time–frequency analysis grid** of the signal (see Fig. 15). Most familiarly, this is the underlying procedure in the *spectrogram* (Potter, 1945; see Fig. 15) and in the *short-time Fourier transform* (e.g., Cohen, 1989). Concatenating the different time frames captures the spectral changes, even though each time frame has a constant time-invariant spectrum (made of time-independent frequency components) that is computed at lower precision due to the limited window duration—each time frame is constrained by its own uncertainty principle that is valid for its modified duration and bandwidth (Cohen, 1995, pp. 44–52). Apparent changes in the frequency content of the system can be gathered from the changes between the short-time spectra over consecutive time frames. In contrast, analyzing the entire time signal using a single Fourier transform would have contained the same amount of data, but at a much higher spectral resolution that would have not enabled us to intuitively appreciate the time-varying nature of the underlying dynamics, because of the unvarying nature of the Fourier frequencies. For example, compare the three spectrograms in Fig. 15 to their respective long-term spectra in the bottom right corner of the figure.

Time windowing is the basis for all time–frequency analysis methods, where signals are taken as joint probability distributions, or energy density functions, in both time and frequency (Gabor, 1946; Page, 1952). In this approach, the signal energy density is defined in time and frequency $p(t, \omega)$, so that the fractional energy in each time–frequency grid point of duration $\Delta t = t_2 - t_1$ and bandwidth $\Delta\omega = \omega_2 - \omega_1$ is (Cohen, 1995, p. 82–92)

$$p(t, \omega)\Delta t\Delta\omega \tag{58}$$

[39]In the context of optics and spatial signal analysis, analogous concepts to time windowing are *aperture* and *pupil functions* and *apodization*. These functions are usually defined in two dimensions and are not always real.

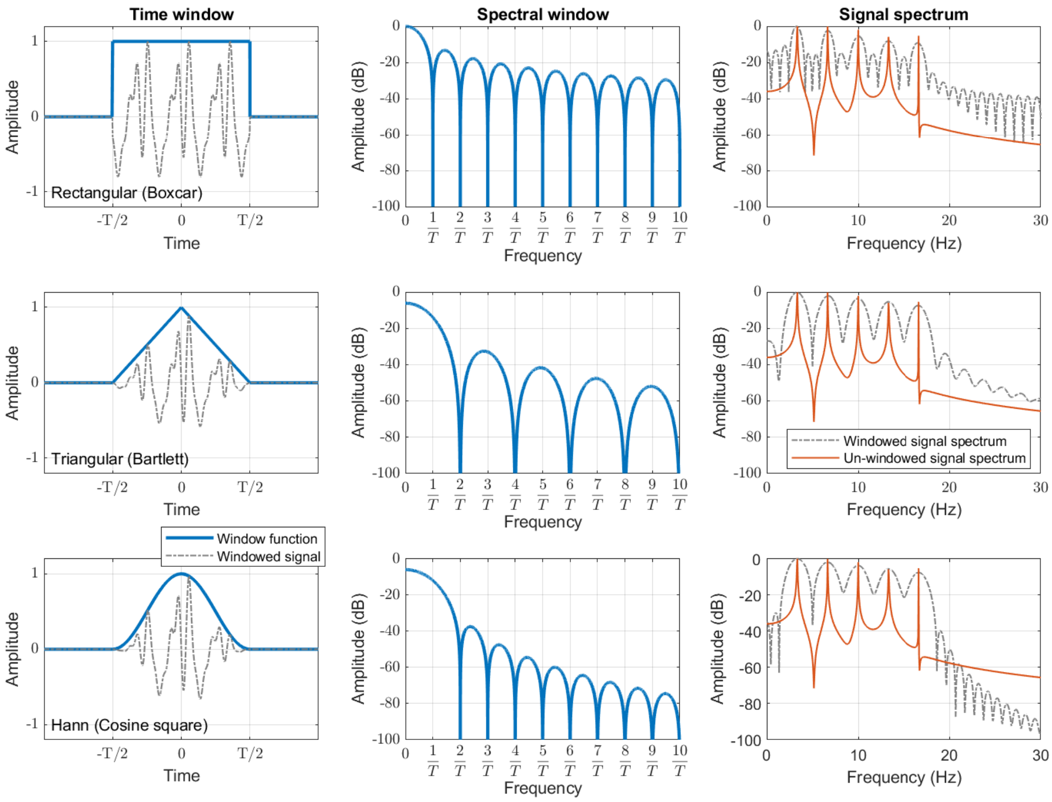

Figure 14: Examples of three basic time windows (left column, solid blue) and their equivalent spectral windows (power spectrum using Fourier transform; middle column, solid blue). A harmonic signal is windowed using the three windows (left column; dash–dot gray) and its FFT spectrum of both the un-windowed version (in red) and windowed one (dash–dot gray) are plotted on the right column, which was obtained through convolution of the spectral window and the signal. The original input signal to all windows has five harmonics $x(t) = \sum_{n=1}^{5} \cos(n\omega t + \frac{2\pi}{n})$, with $\omega = 6.66\pi/T$ where $T = 1$ s is the duration of all windows. All windows are set to 0 at $|t| > T/2$ and their support runs at $|t| \leq T/2$. The rectangular window (top row) is defined as $w(t) = 1$ and its spectrum is $W(\omega) = T\,\mathrm{sinc}(\omega T/2)$. The triangular (Bartlett) window (middle row) is $w(t) = 1 - 2|t/T|$ and its spectrum is $W(\omega) = T\,\mathrm{sinc}(\omega T/4)/2$. The Hann (cosine square) window (bottom row) is $w(t) = \cos^2(\pi t/T)$ and its spectrum is $W(\omega) = T\sin(\omega T/2)/\left[\omega T(1 - \omega^2 T^2/4\pi^2)\right]$ (Shin and Hammond, 2008, pp. 94–100). Large differences can be seen in the frequency content of the output signals that were processed with the different time windows. Note that even the un-windowed spectra of the original signals on the right (in red) suffer from some broadening, due to the finite signal duration and resolution of the FFT algorithm.

By convention, the probability density function $p(t, \omega)$ is normalized, so that the total energy—the integral over the expression 58—is set to unity. The most familiar time–frequency distribution of this kind is the *Wigner–Ville distribution* (Wigner, 1932; Ville, 1948).

Statistically oriented approaches strive to obtain a meaningful power spectrum—normally a long-term stationary measure—within nonstationary processes by various manipulations that can be thought of as conceptually related to time windowing. For example, in the *running transform* of Page (1952), the domain of the Fourier integral is limited between $-\infty$ and t, so it only depends on the signal's past but not on its future. In another approach, the *evolutionary spectrum* features periodic processes that are amplitude modulated with a peak at around t, which produces an effect similar to time-dependent filtering. It results in a local autocorrelation estimate, which is still physically meaningful and almost stationary in the vicinity of t (Priestley, 1981, pp. 821–855). For example, a frame-based autocorrelation algorithm (effectively nonstationary) combined with proper windowing functions and sampling interpolation can be used to estimate the fundamental frequency of speech (Boersma, 1993).

Time windowing is as far as it is possible to apply the traditional, parametric frequency definition to nonstationary signal analysis without making the frequency explicitly time dependent.

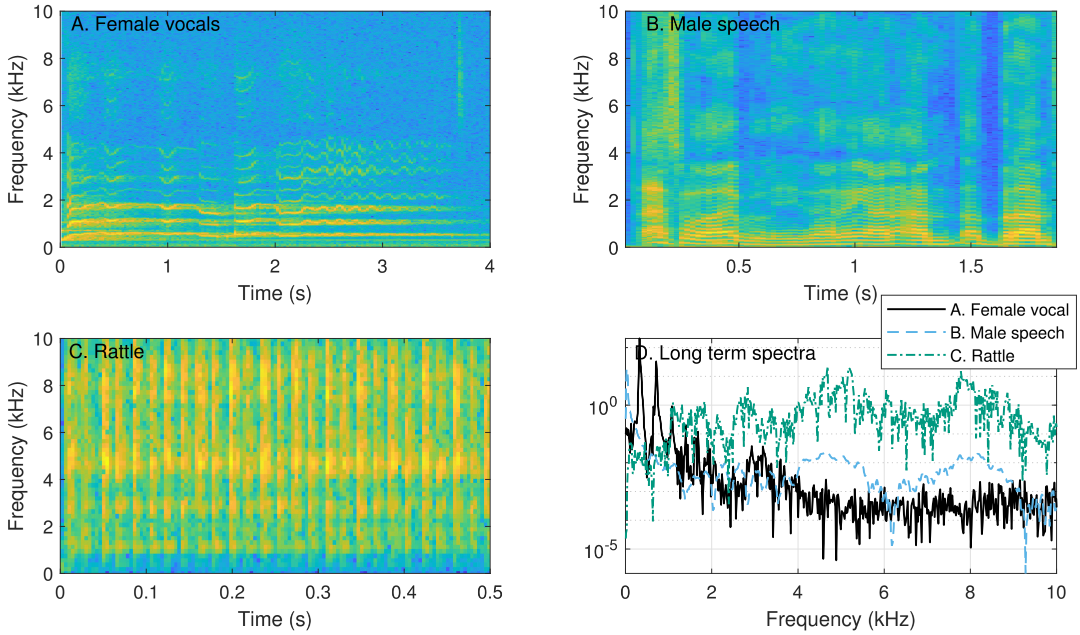

Figure 15: Examples of three nonstationary acoustic signals and their spectrograms, which are a visualization of their short-time Fourier transform. The warmer the color of the time–frequency bin is, the more energy it has. The number of frequency bins and the Hann time-window overlap between processed signal frames was optimized for enhanced overall time–frequency resolution. **A**. Female vocals singing a long "*Love*". Each frame comprised $N = 2048$ samples with 50% overlap between the samples of consecutive frames. The timbre of the voice is determined (also) by the fundamental frequency (the lowest curve) and its harmonics (the parallel curves above it), which move together up and down the musical scale to produce the melody. **B**. Male speech saying "*That's what I believe, I mean, I am... but I'm...*" ($N = 2048$ samples; 50% overlap). Here, the fundamental frequency is lower and has many more audible harmonics, some of which are emphasized by the natural filtering of the larynx and mouth cavity (*formants*). At high frequencies, the sound production tends to be noise-like (turbulent) and a *deterministic* frequency may not exist—only a statistical description of the signal. **C**. A vibraslap sound—a musical rattle that produces a periodic noise-like sound. The periods can be distinctly seen along the time axis, whereas any "pitchiness" of the instrument is much less distinct, as it exhibits a very coarse harmonic structure along the y-axis ($N = 256$ samples; 25% overlap). **D**. The long-term Fourier transforms of the three signals in **A**–**C**, computed using the fast Fourier transform (FFT) with $N = 2048$. It is evident that the temporal structure of the signals is not (immediately) visible in this way, although the same information should be contained in these spectra, ideally (perhaps using a higher N).

3.5.7 Instantaneous frequency I: Definitions

The final stop in the process of decisively differentiating between frequency and time may have been the explicit introduction of time-dependent frequency. Although it was originally introduced as an ad-hoc engineering quantity in radio communication, and despite several associated paradoxes and issues, it is indispensable in communication and electronic engineering and has become a central concept in time–frequency analysis over the century of its existence.

Instantaneous frequency[40] is defined as the time derivative of the phase function $\theta(t)$ (Carson, 1922)

$$\omega(t) = \frac{d\theta(t)}{dt} \qquad\qquad f(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} \tag{59}$$

with $\omega(t)$ being the instantaneous angular frequency and $f(t)$ the instantaneous frequency[41]. Therefore, in the example of sinusoidal FM (Eq. 46), the instantaneous frequency is $\omega_c + m\omega_m \cos(\omega_m t)$, whereas for a standard (unmodulated) sinusoidal signal it is simply $\omega = \omega_c$. The general FM signal

[40] The synonymous term *time-localized frequency* has appeared in recent literature.

[41] The term "*instantaneous frequency*" was used informally already by Brillouin (1914) to designate the derivative of the phase. It was used to describe the initial transient portions of a wave—its *forerunners*—when the wave propagates in an anomalous dispersive medium, before the wave group arrives at group velocity, which generally coincides with the *signal velocity*.

thus takes the form (Carson and Fry, 1937)

$$x(t) = a \exp\left[i\left(\omega_c t + m\int_{-\infty}^{t}\omega(\tau)d\tau\right)\right] \tag{60}$$

in which the argument of the complex exponential is its *instantaneous phase*. For example, see Fig. 16 (left) for the instantaneous frequency of a linear FM (up chirp).

In the probabilistic framework of the signal as a two-dimensional joint probability distribution, the instantaneous frequency is the first frequency moment of $p(t, \omega)$

$$\langle\omega\rangle_t = \frac{1}{p(t)}\int \omega p(t,\omega)d\omega \tag{61}$$

where $p(t)$ is the marginal distribution of the signal with respect to time. This definition entails that the instantaneous frequency is the local frequency average of the signal at time frame Δt (Fig. 16, left and middle). When the signal is stationary, then the local average is equal to the global average, which is then identical to the classical definition of frequency (see Fig. 2 C).

Another definition of the instantaneous frequency that is often invoked is directly derivable from the *analytic signal*—a complex representation of the time signal, whose real part is equal to (half) the measured signal and its spectrum does not contain any negative frequencies (Gabor, 1946). Due to the symmetry properties of the Fourier transform, the real and imaginary parts of the analytic signal are related through

$$z(t) = x(t) + i\mathcal{H}[x(t)] \tag{62}$$

where the operator $\mathcal{H}$ denotes the *Hilbert transform*, which is defined as

$$\mathcal{H}[x(t)] \equiv \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}dt' \tag{63}$$

The integral is evaluated using the *Cauchy principal value* (denoted by $\mathcal{P}$) at $t' = t$. For a rigorous derivation of the analytic signal, see, for example, Mandel and Wolf (1995, pp. 92–97).

There are many advantages for using the analytic signal in time–frequency analysis, where it has become a key tool, along with the Hilbert transform (Vakman and Vaĭnshteĭn, 1977; Cohen, 1995). One of the conveniences in employing the analytic signal is the ability to represent signals in polar form

$$z(t) = a(t)e^{i\varphi(t)} \tag{64}$$

where $a(t)$ represents *instantaneous amplitude* and $\varphi(t)$ is the instantaneous phase of the signal. Depending on the context, both may count as forms of modulation whenever it is possible to define a stationary carrier around which the instantaneous variations occur (see Fig. 2 H). The instantaneous frequency can be obtained, therefore, directly from this expression, by differentiating the argument (the unwrapped phase) according to Eq. 59. For example, see Fig. 16 (middle). *Amplitude modulation* can be factored out as a low-frequency variation around the carrier $a(t)$, which may have its own modulation spectrum that is more conveniently treated separately of the carrier spectrum. If the phase or frequency modulation is also factored out along with the amplitude, then together they constitute complex modulation term (referred to as *AM–FM* in modern time–frequency techniques; e.g., Sharma et al., 2017), which can be employed as a general method of mathematically representing and decomposing arbitrary signals. This decomposition holds either for narrowband signals with a single carrier or broadband signals with multiple carriers, each of which is individually modulated (see Fig. 2 I).

3.5.8 Instantaneous frequency II: Issues

By its very definition, the instantaneous frequency is time dependent and thus any suggestion that it is equivalent to time itself would be logically incoherent. Nevertheless, there are several issues that arise with the definition of the instantaneous frequency and its interpretation, which have contributed to its less than universal adoption and somewhat unclear theoretical status.

The first class of issues with instantaneous frequency relates to the clash with the traditional concept of frequency in physics and the lack of intuition that it garners to complex signals. Already in its introduction, Carson (1922) noted that the notion of variable or instantaneous frequency is difficult to reconcile with our physical intuition of what frequency means. Van der Pol (1946) also

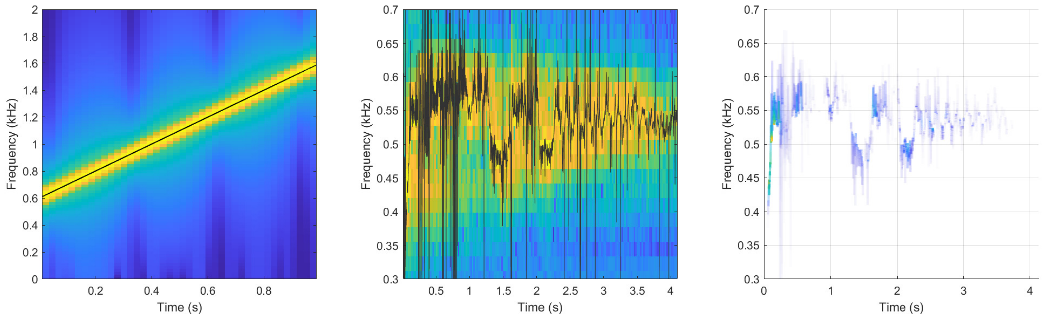

Figure 16: Examples of the estimated instantaneous frequencies of two acoustic signals. **Left**: The first signal is a linear frequency modulation chirp, of the form of $s(t) = \exp(2\pi ft + \frac{d}{\pi}t^2)$ with $f = 600$ Hz and slope $d = 1000$ Hz/s. Using a spectrogram ($N = 2048$, 50% overlap), its instantaneous frequency is blurred both in time and in frequency (color plot). The direct estimation of the instantaneous frequency using the Hilbert transform produces a sharp curve (in black), which directly overlaps the spectrogram. **Middle and Right**: The second signal is taken from the female vocals of Fig. 15 A, where the fundamental frequency was roughly picked using a band-pass filter (fourth-order Butterworth bandpass filter centered at 500 Hz with quality factor $Q = 3.33$). Once again, the colored spectrogram shows the smeared trend, whose center corresponds to the instantaneous frequency, which was also calculated using the Hilbert transform (in black). However, the latter produces very rapid excursions from the mean, which makes it difficult to interpret and be certain of. On the right, an alternative employment of the Hilbert transform is applied to the same filtered signal using the *Hilbert-Huang transform*—a popular applied method to compute the instantaneous frequency of different modes in arbitrary broadband signals (Huang et al., 1998). Unlike the standard Hilbert transform, the instantaneous frequency in this plot is also weighted by the instantaneous amplitude, so the effect of the extremities, which are prominent in the middle plot, is significantly reduced.

underlined the unintuitive nature of the instantaneous frequency compared to the classical frequency concept. As a solution, he analogized it to nonuniform angular motion from classical mechanics, where the *angular velocity* $\omega(t)$ is determined in an identical way to the instantaneous frequency, by differentiating the phase function.

The second class of issues with the instantaneous frequency are those of mathematical inconsistency and uniqueness when applying the particular definition of Eq. 59 in arbitrary cases. Shekel (1953) strongly argued against the usage of instantaneous frequency, at least in its standard definition, since it is not unique for a given signal and its usage is both paradoxical and inconsistent. Mandel (1974) distinguished between the classical definition of frequency as infinitely periodic and that of the mean frequency of a narrowband signal. He emphasized that the instantaneous frequency as the derivative of the phase may produce values that do not actually appear in the measured (Fourier) spectrum. He went as far as to suggest that the two quantities should not be both thought of as frequency, since it produces an unfortunate ambiguity in our analytical understanding. Difficulties arise also when dealing with broadband signals (for example, see Fig. 2 I), which are best expressed as a sum of narrowband signals[42], but may not be amenable to a unique decomposition at that (Boashash, 1992; Sandoval and De Leon, 2018). Even then, the instantaneous frequency may give rise to out-of-bandwidth frequencies, or to negative frequencies even after they were eliminated from the spectrum, and be dependent on the signal remote past and future (Cohen, 1995, p. 40–41). Some of these issues may be a result of the mathematical formalism related to the analytic signal itself (Vakman and Vaĭnshteĭn, 1977), as there is usually a persistent ambiguity regarding a unique representation of the signal, with respect to the allocation of signal variations to the instantaneous amplitude or to the instantaneous phase and frequency (Sandoval and De Leon, 2018). Some of these challenges make the estimation and interpretation of the instantaneous frequency of real-world, arbitrary signals (of the kind of Fig. 2 I) nontrivial. Different methods in time–frequency analysis and signal processing wrestle with this problem, as is illustrated in the example of Fig. 16 (middle, right).

[42]There is no clear-cut definition for a narrowband signal, although it is understood that the modulated carrier and envelope are relatively well-confined to be around one center frequency. In particular, it means that the envelope itself, which generally has its own spectrum, is well-confined to low frequencies that are much lower than the carrier, so the two spectra—the modulation and carrier—do not breach one another (Bedrosian, 1963).

3.5.9 Instantaneous frequency III: Significance

All in all, the concept of instantaneous frequency has not made it into any mainstream signal processing or physics curricula. While applied researchers might still be occasionally grappling with the intricacies of the concept, it is typically omitted from the introduction to the topic of periodic phenomena and harmonic analysis. Its appearances in physics may have been limited to specific problems that tend to be either nonlinear (e.g., Whitham, 1999; Huang et al., 1999)[43] or manifestly modulatory (e.g., Mandel, 1974). It is therefore not featured in any standard introduction to physics, as the Fourier analysis and classical periodicity are—i.e., in the solution of standard differential equations, in the derivation of solution to various physical problems, or in standard signal analysis and processing (§3.4). This implies that the very idea of a time-dependent frequency remains relatively esoteric in mainstream science, despite a prominent role in some communication and electronic engineering applications (see below).

The exception to all this has been the central role that both instantaneous phase and frequency occupy in the study of the nonlinear dynamics of synchronization phenomena in different fields. *Synchronization* is defined as "*an adjustment of rhythms of oscillating objects due to their weak interaction*" (Pikovsky et al., 2001, p. 8). In the generic setup of this problem, each oscillator is autonomous—it contains its own energy source—and its oscillations are time independent, as in the simple harmonic oscillator examples above. However, when two such oscillators are coupled and the difference between their natural frequencies is not too large, their frequencies become gradually detuned until they synchronize. The original discovery of this phenomenon was by Huygens (1665 / 2001), who invented the pendulum clock, and reported that when two swinging pendula hang from the same wooden beam, they synchronize.

To study this rich and complex type of oscillations, it has been instrumental to model them using instantaneous phase and frequency, which often appear explicitly in nonlinear differential equation models of the phase coordinate and its derivatives. Perhaps the best known example of major practical use of synchronization is the *phase-locked loop* (PLL) circuit, which is found in countless many electronic receiver systems, among others, and allows for perfect locking to an arbitrary transmission or a clock signal in a given communication channel (Viterbi, 1959). The PLL contains a local oscillator, a low-pass filter, and a phase detector (a nonlinear circuit component whose output is proportional to the product of the received and local oscillations), and it is connected using a feedback loop, which is essential for the cycle-by-cycle tracking of the external signal by the local oscillator. This system is also sometimes studied within control theory, where the involvement of feedback in the design is a universal design feature of *closed-loop* systems (Abramovitch, 2003). The feedback allows for real-time tuning and error correction toward a specified system response to arbitrary inputs and conditions, by returning part of the output to the input. The common element in all these systems is that they are geared for working in real time—usually around a well-modeled steady-state operation. In general, the knowledge about the past and the future of the signals is immaterial, whereas the proximate behavior within a narrow time window around the present moment is critical to assess its function and performance. Modeling these systems using methods in which the spectrum is strictly time independent, or even smeared due to windowing and statistical averaging, is therefore likely to be self-defeating, whereas the availability of instantaneous quantities is indispensable in analysis.

3.6 Interim discussion

The above review of the concept of frequency loosely followed the historical relaxation of the assumptions that have classically constrained the applicability of the original definition of frequency to strictly periodic oscillations. This has eventually led to the analysis of arbitrary waveforms and signals, including aperiodic ones, using tools and concepts that were developed with periodicity in mind. As frequency is calculated from the time signal periodicity, it is inherently intertwined with time, to the point that the two can seem as one and the same—two reciprocal or conjugate quantities that encompass their own domains. According to traditional thinking, the time and frequency domains are complementary and effectively contain the same information, only in different forms.

What the above review has attempted to prove, though, is that frequency cannot be considered only dependent on the period and it is also not equivalent to time. In its simplest parametric definition, the period and frequency are always dependent on additional non-temporal parameters.

[43] For a short review of select appearances of instantaneous frequency in the form of chirps in physics, biology, and engineering, see Flandrin (2018, pp. 9–20).

In more advanced formulations, a time-independent frequency entails a completely deterministic worldview, which is analytically impractical and epistemically fantastic. Knowledge of the remote past and future also characterizes the statistical approach to signals, which strongly favors working with parametric time and idealized stationarity. With the addition of the Fourier transform to the harmonic analytic toolbox, it has become possible to dispense with strict periodicity, using a one-to-one map between the periodicity axis to that of the time dimension. It has given us access to frequency as a continuous variable, but has also given rise to a potential conflation between time and frequency, or rather, a conflation between time and periodicity. In modern time–frequency analysis, however, a clear, upfront distinction is made between stationary and nonstationary processes, which explicitly makes the time–frequency modeling two dimensional. This is also where the various paradoxes, constraints, and lacunae in the transition between the time and frequency domains are highlighted, whereas a similar, explicit recognition that frequency can be independent of time has not made it into physics or philosophy (see §2.3).

The implications of frequency being independent of both time and space and the possibility that it is a dimension of Reality in its own right are analyzed in the next sections. It should be clarified, however, what we mean by frequency, given the barrage of definitions, nuances, and analytical methods that were mentioned above, which have not yet converged to a universally agreed upon and concise definition. While ideally the instantaneous frequency reduces to the parametric frequency and Fourier spectra in stationary cases, stationarity is rarely met in practice. Furthermore, there is ambiguity with respect to the choice of signal representation, which means that the instantaneous frequency is dependent on the method used to extract it and assumptions behind it. While this is certainly not encouraging when one attempts to grapple with the basic meaning of this concept, we can live with this ambiguity for now and suggest a more qualitative and general definition instead:

Definition 1 *Frequency is a quantity in inverse-time units that is used to abstract patterns of time dependence in arbitrary variables using periodic functions. In its most primitive form, it is limited to the description of cyclical, perfectly repetitive dependence we call periodicity*[44].

As a measure of convenience, we also include time independence and call it zero frequency, despite its abuse of the idea of periodicity. More sophisticated mathematics enables us to use the same cyclical patterns to emulate aperiodic time dependence, without giving up periodicity and its valuable associated tools. Descriptions pertain to approximate periodicity also fall under this definition, by using concepts of probability, statistics, and noise, as applied to ensembles measured over a long time. Here patterns indeed repeat, but their exact repetition instantiation may be unpredictable. Instantaneous frequency is often tied to a *channel*—a central frequency or a carrier—but within this average range there may be no need to commit to exact periodicity, and hence, to repeated patterns. On the other conceptual extreme, we sometimes care about instantaneous deviations from periodicity that can be described using instantaneous frequency—a quantity that allows for much more precise time dependent specification of arbitrary time signals, but at the cost of lost insight about the overall signal evolution and, occasionally, murky uniqueness conditions. In this realm, it is debatable whether frequency describes the patterns in time, or rather, time describes the patterns in frequency.

Our rather loose definition of frequency shall be adequate to further explore the fundamental question put forth in this work: should frequency in some or all of its different forms be taken as an additional dimension of Reality? In simple systems, all frequencies are constant and all associated definitions neatly converge. However, if we do not subscribe to a wholly deterministic view of Reality—or admit that we do not always have access to all the knowledge that is required to analyze a problem (§A.1)—we may have to accept that such neatness applies only to a subset of systems that are encountered in the real world, whereas in other situations a more elusive notion of time-dependent frequency must be at play.

4 Frequency and general properties of the dimensions of (perceived) reality

The analysis in the previous section has attempted to establish that, at least in some cases, frequency can be thought of as a variable that is different from time, despite being tightly interwoven with it.

[44]The definition is made specific for temporal frequency and time, but a completely analogous definition would be correct with spatial frequency and space.

Property	Space	Time	Frequency
Mandatory coordinates	Yes. *Nonlocality may be taken to violate it, in case of quantum entanglement. **Relativistic effects confine the meaningful coordinates to the light cone of the inertial system on which we travel.		Yes, depending on the level of analysis
Movement	In any direction, continuous	Only forward, continuous	Backward and forward, jumps are allowed, depending on the definition
Collisions and interactions	With intersecting coordinates, or overlapping fields	With co-occuring events	When the (carrier) frequencies are nearly the same; interference
Mathematical independence	Yes	Yes	Partially time-dependent. It is down to the uncertainty principle in conservative systems. In non-conservative systems it has some independence from time.
Scalability	Yes	Yes	Yes, but usually interdependent of space and time.
Modulability	Yes	Yes	Yes
Invariance typicality	Yes	Yes	Limited
Tangibility	Yes	Indirectly	More than time, but less than space
Sensory association	Vision, touch	Hearing	Hearing and color vision (temporal frequencies), vision (spatial frequencies), touch (temporal and spatial frequencies)

Table 1: A summary of the nine properties explored in §4.1 and how they apply to the space, time, and arguably, frequency dimensions.

Physically and mathematically, this implies that frequency corresponds to at least one more degree of freedom in the dynamic system. Perceptually, frequency is detected through different dedicated receptors in several sensory modalities, including vision, hearing, and touch, where it gives rise to percepts that are distinct from time and space. The perception associated with physical frequency is also not readily related to the "how often?" question that is classically associated with the simple definition of the frequency as the reciprocal of period.

We would like to go further in the exploration now by asking if the supposed frequency mathematical and perceptual degree of freedom may also be cast as an additional dimension of Reality that is on equal footing with space and time. In order to do that, it will be instructive to elucidate what properties the four dimensions of Reality that are in consensus have that may be generalizable. The following may not be an exhaustive list of properties, but it aims to capture the most key ones that can be applied mathematically, physically, perceptually, and conceptually. Each property listed is explained in terms of space and time and then analyzed also with respect to frequency.

4.1 Nine properties of the known dimensions of reality

The analysis below primarily pertains to a Newtonian conception of Euclidean space and time, which is largely in line with our phenomenological, sensory, and perceptual version of Reality. While taken as a starting point for each property explored, in the extreme cases of very small and very large or fast physics, we have to consult with quantum mechanics and relativity theory, respectively, which can significantly complicate the generalization of some of these phenomenological properties. Nevertheless, the properties listed below are usually general enough to hold at all scales, in spite of occasional strange effects associated with them at the extremes.

When scrutinizing frequency against these general properties, we shall use two primary types of arguments: 1. Straightforward application of frequency to the logic of the known dimensions. 2. A fortiori arguments about the nature of frequency, given the peculiarity of time and the fact that it has already been widely (even if not universally) accepted to be a fundamental dimension of Reality.

The list of dimensional properties is summarized in Table 1.

4.1.1 Mandatory coordinates

Space and time The most fundamental property of space and time, as we phenomenologically perceive them, is that nothing physical (we know of) exists outside of them. In other words, all elements of matter and energy can be associated with a specific region in space and interval in time. This is the same for all actions and events that take place in specific locations and moments. Information, which is less tangible, is also physical and must be stored somewhere (Landauer, 1996). The location and duration of all these things can be therefore represented using particular coordinates—either points in space and moments in time—or using zones defined by coordinates—regions in space and durations in time that are occupied by continuous objects and events.

According to the classical Newtonian point of view, there is a distinction between *relative* and *absolute time and space* (Newton, 1687 / 1999, Definition 8, Scholium). Habitually, we reference time that is external to our own body—following the motion of moving bodies—that gives rise to a relative and approximate measure of time. However, there exists absolute and true time that is in uniform flow and does not depend on anything external to it. Similarly for space: it is immovable, homogenous, and absolute and does not require any external reference, although we can talk about motion relative to it.

Both the quantum and relativistic points of view significantly complicate the validity of this classical conception of space and time. Quantum physics is characterized by various *nonlocal effects*, which entail the dependence of a localized measurable quantity on something that occurs in remote, noncontiguous points in space. However, even the negation of classical space and time dependence is still logically defined through the very concepts of space and time, so it is difficult to altogether eliminate these dimensions by considering nonlocal effects. See further analysis in §8.

On the other extreme, both special and general relativity theories advise us that there is no meaning for absolute coordinates, given the impossibility to have an agreed upon "now" moment between moving objects at velocities approaching the speed of light. The very concept of "now" loses its intuitive meaning, because space and time cannot be separated and everything exists in a four-dimensional spacetime. The speed of light is invariant in all systems, but spacetime depends on the system's own velocity. However, the choice of an inertial reference frame is arbitrary as the laws of physics are identical in all of them according to the *principle of relativity*[45]. According to general relativity, spacetime does not exist independently of matter and the gravitational field that is formed as a result (Einstein, 1954). However, if we entertain the idea that spacetime has been given rise to with a particular metric[46], then we nevertheless must use some coordinates—relative, ad-hoc, local or others—to describe the physics of all moving objects. Here, the best that we can do is to confine our relative coordinate system to the light cone on which our inertial system travels, in order to retain its meaningfulness, which would otherwise be lost between non-intersecting light cones[47]. In large-scale precise time measurements the time coordinates of different reference frames of interest are typically referenced to common astronomical events that form agreed-upon points in time, which serve as absolute markers that are only valid for the selected coordinate system (Audoin and Guinot, 2001, pp. 16–37). In spite of this, while undoubtedly troubling, these conceptual limitations on the very meaning of having a common coordinate system are acceptable for the phenomenological and perceptual perspectives in the present context (but see §9.9.1).

It is possible to abstract certain processes and models from this binding spacetime framework (for example, in pure mathematics and statistical analysis), or from time only (by setting it as a parameter). Despite these mathematical representations, they do not necessarily entail that anything can exist outside of spacetime, or at least outside of 3D space. Once an arbitrary reference point in space and time is chosen, every object and event may be associated with coordinates, or a region within spacetime, to within some degree of uncertainty. In analysis, the mandatory coordinate property of the dimensions has been used to express the dynamical equations of all mechanical, electromagnetic, and quantum systems, which are formulated using the observed functions and their derivatives in space and time.

Frequency Can frequency be considered a mandatory coordinate for any element of matter or energy? A different way to ask this question is whether there exists anything in the universe that does not vibrate or oscillate, or that cannot be associated with a frequency or a spectrum.

In macroscopic systems, object frequencies can be associated with rotations around axes in three dimensions or with vibrations along the axes. Rotations are often modeled in mechanics as occupying

[45] It has been suggested that there is a *preferred reference frame* which is determined by the small anisotropy of the cosmic background radiation (Land and Magueijo, 2005). However, the choice may still be a matter of convenience for the observer—to work in a reference frame within which the cosmic background radiation is isotropic—and is not an absolute one (Melia, 2022).

[46] A *metric* is the mathematical function that defines distance in spacetime, which is different between classical (Euclidean) space, special relativity with spacetime in Minkowski metric, and general relativity with curved spacetime.

[47] That is, physical systems that have drifted so far apart in spacetime that they may never be able to meet again.

their own three degrees of freedom, apart from the standard three dimensions, and all together they may be considered the *generalized coordinates* of the system in a mathematical "*configuration space*" (e.g., Goldstein et al., 2014).

On the quantum level, all particles are associated with particular frequencies given by two basic relations that are universally applicable: the *Planck relation* for photons

$$E = hf = \hbar\omega \tag{65}$$

where $h = 6.626 \cdot 10^{-34}\ J \cdot s$ is the *Planck constant* and $\hbar = h/2\pi = 1.054 \cdot 10^{-34}\ J \cdot s \cdot rad^{-1}$; and, the *de Broglie wavelength* for *matter waves*

$$p = \hbar k \tag{66}$$

where k is its wavenumber $k = 2\pi/\lambda$, and p is the momentum of the particle moving at group velocity. For quantum objects as small as diatomic molecules there are additional oscillatory quantities that are analogous to those found in macroscopic motion, whose corresponding line spectra predominantly depend on contributions from the quantum vibrational and rotational energy levels.

In relativistic physics, frequency is generally a latent degree of freedom. It may be made explicit in dispersive media, through the speed of light that is a constant (frequency-independent) only in vacuum where $c = \omega/k$, but is generally frequency-dependent in all other media as $k = k(\omega)$. Just as in classical mechanics, time measurement in relativistic systems still employs frequency-based clocks, whose periods are themselves subjected to relativistic transformations between frames, which have to be taken into account in large-scale measurements (Audoin and Guinot, 2001, pp. 16–37).

The generalized notion of frequency afforded by the Fourier transform entails that even aperiodic structures can be expressed using periodic functions. It also associates constant, unvarying functions with zero frequency. Problematic as it may be (§3.4.2), we can invoke this classical framework to attach frequency coordinates (values or regions from the Fourier spectrum) to all matter and energy distributions. Uncertainty and limits to deterministic knowledge would then ensure that it is at least partially independent of the time coordinate. Most generally, the notion of instantaneous frequency (§3.5.7) can be used to describe time-varying functions, where the frequency itself may be changing nonstationarily. This may be especially handy in a local sense that does not appeal to deterministic knowledge of the associated spectrum.

Perceptually, all of our modalities that interface with the external environment are frequency dependent, either directly through the sensory organ filters (vision, hearing, touch, balance) or indirectly in all other senses, including those that may be non-spectral (olfaction, gustation, pain, etc.). An indirect frequency sensation can be attributed to any sense once we apply time–frequency analytical tools to the functions that describe the stimuli or their sensed response. For example, taste is not normally associated with frequency, and yet the sensation of sweetness as a function of the spatial-temporal concentration of sugar on the tongue can be characterized as a time function (Travers and Norgren, 1989; Iannilli et al., 2014), which is generically amenable to time–frequency analysis. This frequency-dependence—likely an aperiodic one—may be more suitable to express the modulation domain of the signal rather than a carrier frequency per se.

In summary, there is no difficulty to assign a frequency coordinate to any physical variable that is characterized by space and time. In the simplest cases, the frequency is either reduced to a constant parameter, or is assigned the 0 Hz value, without loss of generality.

4.1.2 Movement

Space and time The basic dynamic property of space is that material objects and radiation of any kind can move about the geometry spanned by the three spatial dimensions, as long as the path is contiguous (i.e., without jumps[48]). As for time, movement appears to be both contiguous and restricted to one direction—only from past to future—despite numerous works of fiction that dispensed with this limitation (Gleick, 2016), beginning with Wells (1895). This produces the fundamental relationship between cause and effect, where the former must precede the latter in order to comply with our understanding of Reality (Poincaré, 1913 / 1976). However, at high velocities, the relative speed in which there is movement to the future does appear to vary between observers moving at different velocities than the moving object they observe. The relativistic "*proper time*" of the moving object—its own clock that is different than the observer's clock—captures this difference.

[48] Note that even the hypothetical wormholes of the general theory of relativity only allow for *apparent* jumps in Euclidean space due to extreme local features in spacetime topology produced by its non-Euclidean metric (e.g., Morris et al., 1988).

Frequency What can the meaning be of "movement in frequency"? "Moving about" in frequency is qualitatively different from moving in the spatial dimensions and is unlike the unidirectional movement in time. The answer to this question depends on the frequency definition that is being looked at. The Fourier spectrum is by definition time invariant—each frequency component in the spectrum is infinitely long and can be thought of as inertial if taken in isolation (with no apparent cancellations by other frequencies). The totality of (infinitely) many such inertial components gives rise to dense spectra that can appear as frequency-varying in the time domain (Fig. 12). Despite its mathematical correctness, this solution seems to be missing the point. A more insightful vantage point may be to look at the general decomposition of signals to carrier and envelope terms (Eq. 64; see example in Fig. 2 H). In the simplest of cases, there is a high-frequency carrier, or a mean frequency, which remains fixed and all spectral changes in time can be associated with the slow-varying complex envelope around the carrier. But this decomposition is not unique and it may be difficult to pinpoint as for where the change lies—is the normal mode (associated with the carrier) being changed, or only the force that impacts it (associated with the envelope)? More complex systems contain multiple normal modes (i.e., frequency components or carriers), which tend to have an even more ambiguous decomposition (e.g., Fig. 2 I). These systems are generally not continuous in spectrum, which is concentrated around the carriers and exhibit "spectral holes" between them. Movement in frequency in these cases may be complex and not uniform across all modes, so multiple trajectories in the frequency dimension may be required to describe it. Despite this marked ambiguity and high degree of complexity, there is no conceptual difficulty in associating spectral changes with particular frequency components, which may then appear to be moving like objects in space, at least locally.

It is perhaps instructive to make a distinction between measurable spectral changes that are inertial versus those that require energy transfer into or out of the system. This is because time–frequency analysis alone may not be able to distinguish between the two without additional information about the system and its boundary conditions. For example, the classical Doppler shift effect can be measured as frequency modulation of radiated light or sound by an observer relative to a moving source, even if both observer and source are inertial in their own systems. For the static observer, the moving object may well count as entering its otherwise static system and injecting energy into it. If the system is taken to enclose both observer and source at all distances, then the total energy is constant and the entire movement can show in the time-invariant Fourier spectrum as numerous spectral lines, as in linear frequency modulation, for example (Fig. 12; see § 3.5.1 regarding the inclusion of all forces in the problem).

In another instructive example, if we return to Van Der Pol's analogy between the instantaneous frequency definition and angular velocity—both being the derivative of a phase function with respect to time (Van der Pol, 1946; see § 3.5.8)—then the spectral interpretation of the time-dependent motion of a planet in an elliptical orbit around a star (i.e., with variable angular velocity) may be a puzzling case, since there is no net energy transfer there between the star and the planet and conservation of energy is maintained by instantaneously varying radial and angular velocities (Goldstein et al., 2014, pp. 70–127)[49]. Therefore, in this case, the inertial decomposition offered by the Fourier analysis may be much more intuitive and correct, as any apparent modulation in the observation is fully accounted for by all the observable forces, all of which are conservative. Therefore, we may choose to not register any movement in frequency in this system.

Unlike movement in space and time, frequency jumps may be possible (i.e., between two frequencies $f_1 \neq f_2$) without having to sweep across all the values in between the two. It is the standard observation in the spectrum of quantum transitions between energy states, although a recent study suggests that the transitions are of finite time duration and they follow a *deterministic* spatial path (Minev et al., 2019)[50]. Macroscopically, frequency jumps are possible in every modality when a generator is swapped or modulated quickly (e.g., a loudspeaker may produce two well-separated tones with no detectable sweep or non-tonal noise between them).

In summary, spectral movement is possible and common and is unrestricted in its direction. Frequency jumps appear to be just as common, unlike jumps that are prohibited in the spatial and temporal dimensions.

[49] It is not customary to look at the rotation spectrum of planets or talk about their instantaneous frequency, but rather about their average periods and time-dependent angular velocities. However, in the case of the solar system, the various periods associated with the sun, moon, and Earth form the basis for timekeeping and dating, so in that sense, these periods are not entirely different than those provided by man-made clocks, where referring to the clock frequency is customary.

[50] However, no time–frequency measurements were reported in that study. See §A.2 for a more detailed discussion.

4.1.3 Collisions and interactions

Space and time For objects and fields that overlap in their coordinates or are positioned within reach of a certain far field, it is expected to observe some kind of interaction (object–object, object–field, or field–field). Depending on the specifics, these include attraction and repulsion (scattering), collisions, deformations, phase transformations, chemical and nuclear reactions, and others.

Frequency *Interference* between two waves is observed when their carrier frequencies are either identical or very close (it is a given that the waves overlap in space and time). There is some associative resemblance here to interaction between rigid objects in three dimensions: interference may be thought of as an extension of the concept of collision into the spectral dimension, where the impact depends on the frequency (as well as phase and amplitude) difference between the waves and the final interference product may not resemble the input waves, constituting an interaction effect. This aspect of the 5D representation that includes frequency was alluded to by Wiener and Struik (1928), who suggested that relativistic wave coherence (i.e., cross-correlation of two waves at "*nearly the same frequency*") could be explained more readily through the addition of an extra phase dimension to the quantum wave functions, in line with the 5D theories of Kaluza (1921) and Klein (1926).

More complex interactions between frequencies in different modalities may be possible in the context of special phenomena, such as the acousto-optic effect, or the piezoelectric effect. Unlike interference, these interactions generally lead to modulation and energy transformation between waves, so they are observable at a much broader range of frequencies than interference, as long as the interacting waveforms overlap over the same spatial and temporal coordinates.

4.1.4 Mathematical independence

Space and time Mathematically, quantities that take up their own dimensions cannot be expressed using other dimensional quantities alone. Thus, each dimension holds some information that is not found in the other dimensions. Realistically, however, quantities that manifest within the spatial and temporal dimensions are often interdependent, so (for point objects and quantities) there may be fewer degrees of freedom than dimensions, due to various constraints that tie the dimensional dependencies together. For example, in certain mechanical problems (e.g., in the central force problem) this enables the parametrization of the trajectory using time—effectively eliminating one variable / coordinate / dimension from the solution.

Frequency When it comes to frequency, its interdependence with time is very high in conservative systems, but even there it is not total, as was argued throughout §3. It is not possible to arbitrarily specify a signal both in time and frequency without some constraints applying. The uncertainty principle is one such constraint that is most evident with very narrow distributions in time (duration) or frequency (bandwidth) (§ 3.4.6). Cohen (1995, pp. 127–128) discusses the concept of *signal representability* (or *realizability*), where although arbitrary two-dimensional time–frequency distributions can be mathematically specified, they may not correspond to any realizable signals in actuality.

Another fundamental physical constraint ties the relation between the temporal and spatial frequencies for a particular medium. This goes back to dispersion—the dependence of the wave velocity in the medium on frequency (§ 3.1.3)—a constant in vacuum for electromagnetic radiation, nearly constant for light frequencies in air, but variable in most other conditions. Notably, audio-frequency sound wave velocity at standard atmospheric conditions is nearly independent of frequency (dispersionless) as well, at least for short distances and relatively low frequencies (Vigran, 2009, pp. 122–124). In all other media, some dispersion should be assumed for all types of wave propagation (Brillouin, 1960). The dispersion relations of Eqs. 16 and 17 are defined by medium parameters such as material composition, density, and structure. In some regions those properties are time dependent as well. This makes the dimensions interdependent in a complex way, possibly leading to the number of degrees of freedom (e.g., per particle) to be smaller than the number of dimensions. This is visually summarized in Fig. 17, titled somewhat bombastically "*the frequency accessibility paradox*," with the intent to underscore that frequency (be it a dimension or other) cannot be completely disentangled from time and space.

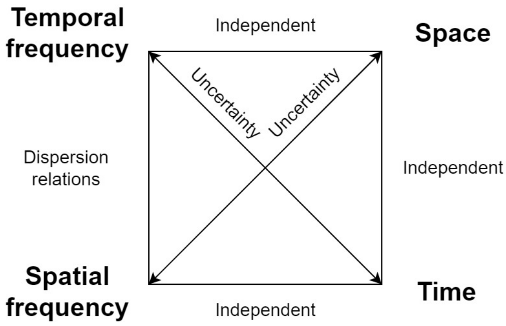

Figure 17: The frequency accessibility paradox in classical and quantum systems illustrates the minimal constraints that apply to the five quantities in all physical systems: space, time, and frequency. More constraints may apply to further reduce the degrees of freedom. In the case of quantum systems, the temporal frequency of photons is proportional to the energy (Eq. 65) and the spatial frequency of matter waves to the momentum (Eq. 66), while independence implies that the quantum operators associated with these quantities (as observables) commute, while uncertainty implies they do not commute.

4.1.5 Scalability

Space and time Objects both in space and in time are scalable. There is neither a mathematical nor a conceptual difficulty to stretch or compress them either spatially or temporally, although it is not always physically and practically feasible. The fabric of spacetime itself seems to be continuous, so that scaling objects within it does not lead to odd discretization effects, at least not on a macroscopic level of observation. Also, for every intent and purpose, both space and time are of the same size as (or they define the size of) the universe itself, so in practice one does not run into a ceiling effect as a result of overstretching objects in spacetime (i.e., an object will never be larger than the universe that contains it). On the quantum level, discretization effects (and subsequent limitations on observations) of the order of the Planck constant—the *Planck length* $l_p \approx 1.616 \cdot 10^{-35}$ m and the *Planck time* $t_p \approx 5.390 \cdot 10^{-44}$ s—have been theorized (Planck, 1899; Peres and Rosen, 1960; Mead, 1964; Tomilin, 1999). If correct, these would lead to respective discretization effects on scaling, and specifically to floor effects for hypothetical smaller geometries.

Frequency Macroscopic frequency is scalable, depending on how it is generated. Mathematically, the frequency range may be infinite and real, so there are no minimum or maximum values that it can take. Physically, though, the frequencies are sometimes defined to be strictly non-negative (see Footnote 25). Additionally, there may be an upper bound to how high the frequency can be. In vibrational systems, the spectrum becomes compressed if the spatial and temporal dimensions of the system are stretched (and vice versa—a compressed or stretched spectrum suggests a respective change in the spatial and temporal dimensions). Here, frequency is constrained by the other dimensions, yet in order to be fully determined it depends on additional extra-dimensional parameters of the system (e.g., medium density, elastic properties). Note that to the extent that the modulation and carrier frequency ranges can be distinguished from one another, they are usually associated with distinct functions of space and time, and hence with a distinct spectra that may not be independently scalable—all depending on the spatial and temporal sources of the carrier and modulation domain functions. On the quantum level, the energy states of bounded quantum systems (unlike free particles) are generally determined by combinations of constants and integers, which are not as malleable as macroscopic parameters are. Thus, free scaling is generally unavailable here due to discretization. A continuous frequency scale from a discrete frequency spectrum is then obtained through different broadening effects that are applicable to larger systems.

In summary, macroscopic frequency is scalable, but in a way that is generally interdependent on the spatial and temporal scaling properties of the system. On the quantum scale, energy level discretization tends to limit the scope of frequency scalability.

4.1.6 Modulability

Space and time Every force or parameter that is associated with the dynamics of a wave (or a signal) may be spatially and / or temporally varied through modulation. In general, since the wave is defined over space and time, modulation over either one of the spatial or the temporal dimensions would necessarily have an effect on the other dimension(s).

Frequency Frequency may be directly modulated, as is commonly done in radio communication, acoustics, and music, among many other domains. For instance, different musical instruments employ various methods of modulating their pitch either continuously or discretely, to produce certain timbral, melodic, and harmonic movements in the music. In hearing research, it is common to measure the perceptual response to *spectro-temporal modulation* that is defined over both the temporal and the spectral dimensions of the signal (Aertsen et al., 1980b,a). In vision and optics, changes of the carrier frequency[51] of the optical objects generally lead to changes in perceived color, as is encoded by the visual system (e.g., Land and McCann, 1971; Land, 1977). Changes in the spatial frequency content of the optical object relate to how its coarse or fine details appear, as can be predicted from the modulation transfer function of the imaging system, that is the eye (Goodman, 2017). The combination of both temporal and spatial modulation has been studied as *spatio-temporal modulation* transfer functions in vision (Van Nes et al., 1967).

All in all, disentangling the different dimensional contributions of any modulation may be somewhat contrived, since in practice, few (if any) changes to the signal can be made to manifest only one-dimensionally.

4.1.7 Invariance typicality

Space and time As the substrate of physical existence, space tends to be remarkably tolerant to any change in the absolute coordinates (*homogeneity*) and to the direction of movement (*isotropy*). A similar property is the tolerance to the changes in absolute time coordinates (stationarity, § 3.5.3). Put differently, numerous phenomena appear to be both time-invariant and space-invariant (translation-invariant). Effects of memory and nonlinearity locally disrupt these invariances, but most events and systems seem to be indifferent to where they are positioned in the universe, as long as all the relative relationships to the various surrounding media, fields, and forces are equal between the positions.

Frequency There are many situations in which frequency does not have a direct effect on the wave dynamics, which is captured by *the geometrical approximation* that is regularly used in both optics and acoustics (Born et al., 2003; Morse and Bolt, 1944). In this case, the only frequency effects, such as applying to the velocity in the medium (dispersion), amplitude in the medium (absorption), or to surface reflectance, would be those dictated by the medium properties. Here the wave phase at the image plane may be neglected as it is the intensity that is being detected—a power spectrum type of image (§3.5.3), rather than an amplitude image that retains a *deterministic* phase structure. This well-describes situations in which the wavelength is much shorter than the relevant spatial boundaries of the system. For instance, the incident sound between the orchestra to the audience, as well as the various reflections from other surfaces in the space, constitute relevant spatial distances in the acoustical design of concert halls, which are much larger than the longest wavelengths produced by the majority of musical instruments. Spectral invariance may also apply to musical melodies, which may be transposed to a different scale or register—a relative change in frequencies. In this case, it would still retain its melodic identity, which is determined by the relative intervals between the notes in the melody, and their respective durations. Perceptually, though, transposition only applies as long as it is within the melodic range of human hearing (Attneave and Olson, 1971; Pressnitzer et al., 2001).

Spectral invariance appears to break down more often in common situations than do spatial and temporal invariances. A strong spectral dependence is found with various interaction effects that are exclusive for certain frequencies or wavelengths, which in analogy could be compared to very crowded regions in space, or a large density of events (in time). For example, molecular spectroscopy is concentrated on mid-infrared frequency with the Raman *fingerprint region*, loosely defined to be at 1300–900 cm^{-1} (e.g., Fiore and Pellerito, 2021). Such a molecular spectrum cannot be thought of as relative, since its absolute frequencies determine the very identity of the molecule.

In conclusion, while spectral invariance can characterize many classical systems, parts of the electromagnetic spectrum have unique interactions that render many systems spectral variant. Also, as all sensory systems are bandlimited, any spectrum invariance within a particular modality is of limited extent.

[51] Within the visual range, it is typical to refer to the wavelength of the light waves, rather than to their frequencies. Most references to frequency in vision relate not to the temporal frequency of the carrier, but rather to spatial frequencies, which are used for the description of the geometry of the optical object.

To the above seven properties, we shall add two additional ones that are more narrowly related to our perceptual experience as humans and possibly to at least some of our non-human animal relatives.

4.1.8 Tangibility

Space and time *Tangibility* refers primarily to the property of objects that can be perceived by touch—material objects that evoke tactile sensations when touched. Somewhat more ambiguously, tangibility also refers to the property of being perceived by the senses. It is defined as: "*real and not imaginary; able to be shown, touched, or experienced*", or "*a real thing that exists in a physical way*" (Cambridge Dictionary[52]). Or, "*capable of being perceived especially by the sense of touch: palpable*", or "*substantially real: material*", or "*capable of being precisely identified or realized by the mind*" (Merriam-Webster Dictionary[53]).

A dimensional perspective on the concept of tangibility would ascribe it to the space that the objects occupy. Arguably, solids are more tangible than liquids, whereas gases may be altogether intangible, especially if they are colorless and odorless. Also microscopic objects the size of microbes or smaller are not amenable to touch, and macroscopic objects that are too large can be touched, but their full size cannot be truly appreciated (like a wall, a mountain, or a planet). In all other cases, the information combined from the touch and visual modalities is often consistent and complementary, so what looks tangible is indeed tangible, and what feels tangible is generally visible.

In contrast to the spatial attribute of the objects, the time dimension is not directly tangible—only indirectly, through the understanding of dynamics and cause and effect and how objects change as a result. Objects that continually change in time may be perceived as lacking in tangibility if their properties cannot be confidently pinned down. Purely auditory objects are also intangible if they are not accompanied by inputs from other modalities (e.g., Schraffenberger and van der Heide, 2015).

Frequency Is frequency tangible? Yes and no. If tangibility relates exclusively to touch, then the effects of frequency are certainly felt across space and time. For example, the spatial frequency spectrum of objects relates to their contour and texture. Their temporal frequency content relates to felt vibrations upon touching. Touching an object dynamically (stroking, rubbing, hitting, etc.) produces a stimulus that combines its spatial and temporal frequencies. Sound is not tangible per se, but has no meaning without frequency or pitch (even if perceived as pitch-less, as in the case of white noise). And in vision, frequency gives us color, which is not a property that can be felt by touch either. All of these are no less tangible than the time dimension, but they are less tangible than the spatial dimensions, which are inseparable from our senses of positioning and movement of objects.

4.1.9 Sensory association

Space and time Spatial coordinates are most immediately associated with vision and touch (see Fig. 1), which elicit the effect of tangibility. Time is much more abstract than space and we become conscious of it as a supra-modal percept that is not peripherally detected with any one sense. The passage of time has been most strongly linked with hearing (Weisser, 2021, p. 6), yet stimuli to all senses have indispensable temporal as well as spatial attributes, which become mandatory in eliciting the actual perceptions and the resultant information that maps the objects in the environment (Fig. 1).

Frequency As was noted in §2.1, both vision and hearing are strongly associated with frequency. Hearing is primarily associated with temporal frequencies—the frequencies that determine pitch, timbre, harmony, and melody. Temporal frequencies in hearing can double up both in the carrier and in the modulation domains, which can sometimes have complex interrelationships. Slow modulatory frequencies determine level changes, rhythm, beating between adjacent tones, etc. Vision,

[52] https://dictionary.cambridge.org/dictionary/english/tangible, accessed 30.11.2023.
[53] https://www.merriam-webster.com/dictionary/tangible, accessed 30.11.2023.

in contradistinction, is more clearly split between temporal-carrier and spatial-modulation frequencies. Colors—the percepts stemming from the broad tuning of photoreceptors to three (in standard human vision) different ranges of the electromagnetic spectrum—are associated with the temporal frequencies of the light spectrum from the objects (usually discussed in terms of wavelengths), factored as carrier frequencies. The objects themselves are often defined using spatial frequencies, which produce the go-to modulation spectrum when visual images are analyzed and processed (Duffieux, 1946 / 1983; Goodman, 2017). As was discussed in §4.1.8, the object surface can be thought of as a particular configuration of spatial frequencies, which when dynamically moved, transform to temporal frequencies. If the object internally vibrates, then the vibrational frequency is temporal and associated with the carrier domain, whereas the textural frequencies more readily belong to the modulation domain. Although olfactory detection does not seem to be based on spectral principles (see Footnote 3), as long as different substances can be uniquely identified using their vibrational spectra, their spectrum becomes a relevant parameter in objectively characterizing olfactory stimuli. Other senses are more narrowly designed to target very specific types of objects, where frequency may not be a key attribute.

4.2 Frequency as a dimensional property of Reality

Ultimately, the above analysis merely supplements the one in the previous section (§3), given that we only have two kinds of dimensions from which to infer the properties of a general physical dimension, which may be insufficient. Space and time do not behave identically, and frequency too does not exactly follow these general properties in an identical manner to either space or time. The alternative possibility—that frequency is an important quantity that is not a dimension in its own right—may be considered in case that frequency markedly stands out with respect to one or more of the nine properties we listed.

Three dimensional properties listed above may be considered odd when applied to frequency: movement (§4.1.2), mathematical independence (§4.1.4), and invariance typicality (§4.1.7). Movement in frequency stands out, because unlike space and time, it appears that jumping between frequencies in a discontinuous way is possible. This may be due to quantum effects, but can also apply to classical systems, depending on the definition of the frequency source. However, time too is subjected to a unique rule of movement—it can move in a single direction only—at least according to current knowledge. Thus, this may not be a significant oddity: each dimension type has its own unique movement rules.

Frequency also behaves differently with respect to its mathematical independence. As was argued throughout §3, it is an elusive thing to demonstrate, mainly because of the interdependence that frequency has with time. In many physical models, frequency is absent and can only be made explicit through the inclusion of dispersion or other spectral dependences of the parameters. This is different from space and time, whose roles tend to be mathematically explicit. Once again, it may not be a significant difference in its own right to disqualify frequency from being a dimension, but rather a unique feature that it has, which may have historically led to its elusiveness.

The last property that stands out—that of systems often not being frequency-invariant—may be the most interesting one, because it reflects many of the properties that make our reality the way it is. These spectral "islands" correspond to phenomena that are specific to particles, atoms, molecules, object sizes and shapes, duration and progression of events, etc. In many cases, our senses are tuned to receive information at these frequencies and not in others, in a way that ends up being perceived uniquely (as color, sound, touch, etc.). A shift in these frequencies cannot be made without affecting the entire cascade of physical, chemical, and biological filters that depend on the absolute values of these frequencies. Whether this interaction between frequency and (perceived) reality is a cause for disqualifying frequency from the dimensional count may, in the end, be a philosophical choice. We argue that this is what makes frequency special, as it ultimately leads to answers to the "What" question, just as space specializes in answering the "Where" and time in the "When" questions.

5 Synthesis

The case for frequency as a mandatory dimension of Reality has been argued for above. While the idea of including a complex concept such as frequency in the standard count of dimensions may come across as an abstract imposition, it was shown that logically, perceptually, physically, and mathematically, excluding frequency would be inconsistent with important applications in modern science and engineering, as well as situations of ordinary perceptual experience, where instantaneous spectral events are key. According to the analysis in §3 and §4, the exclusion of frequency as a dimension may only be logically justified if either

1. Time is rejected from the standard count of the obligatory dimensions of Reality, so that there is no inconsistency between how we treat time and frequency, or
2. The universe is fully deterministic with total knowledge of past and future, so that frequency can be retained either as a parameter or as a dependent variable.

Both propositions carry substantial metaphysical weight that may not stand to reason with the normal intuitive, phenomenological perception of Reality, or with standard physics. This does not mean that they are impossible, but rather that they may apply in some cases, but not universally. Alternatively, they constitute a step backwards from some of the findings established earlier in this work. Namely, acceptance of either 1 or 2 would entail: dismissal of the perceptual experience of time-dependent frequencies, no level of uncertainty exists of time signals and their associated spectra at any scale—they exist and can be known deterministically, and rejection of the very notion that time can be its own dimension within physics. All that said, while we may reject 1 and 2 as not corresponding well to our standard perceptual and experiential reality, we must acknowledge that they do represent possibilities that exist notwithstanding and may be more applicable for some physical systems (if only through modeling), although perhaps not universally. This reasoning is distilled into the following theorem, which is a corollary of all of the above.

Theorem 1 *Only one of these three propositions can be simultaneously true:*

P1. Time is not a fundamental, obligatory dimension of Reality.

P2. The universe is fully deterministic with total knowledge of past and future.

P3. Frequency is a fundamental dimension of Reality.

The three propositions may be immediately interpreted as referring to 3D, 4D, and 5D conceptualizations of Reality. The theorem is depicted graphically in Fig. 18.

The choice of wording in the theorem "*only one can be simultaneously true*" should be clarified with an allusion to Einstein, Podolsky, and Rosen (1935), who stated that two quantities that are tied through the uncertainty relations (such as quantum position and momentum) "*cannot have simultaneous reality*", which entails that observing one precludes the observation of the other. Similarly, the three propositions of the theorem above are mutually exclusive, in a logical sense. But we would like to underscore that the particular proposition that is in effect need not be permanent, as might be implied by the logical relation alone. This would normally be worded by stating that "only one of the three propositions can occur at one time". But since the propositions themselves directly frame the existence of time, this seems to be circular, as though the choice takes place in time but is outside of time. The concept of simultaneity usually refers to things that happen together in time as well. But the word "simultaneous" is also defined as "*satisfied by the same values of the variables*[54]." Therefore, "simultaneous" seems somewhat more appropriate and less committing in this context, although it results in this odd wording. Finally, we cannot adopt the wording of Einstein et al. (1935) verbatim, so that "P1–P3 cannot be simultaneously real", because then we would be talking about a "real Reality".

The theorem is deducible from the synthesis of the concept of frequency as was presented in sections §3 and §4. A shorter logical proof of this theorem is as follows and others are likely possible. In this proof we take physics and physical systems as a sample representation of Reality—and in turn—of the universe (but see §5.1). Consult §A.1 for the Laplacian definition of determinism as is used in the present work and the applicability of the Fourier integral to arbitrary problems[55].

[54] Merriam Webster Dictionary, accessed 11.1.2024, https://www.merriam-webster.com/dictionary/simultaneous.

[55] The following proof depends to some extent on resolutions to a host of long-standing metaphysical questions that are far from being universally agreed upon: the meaning and definition of determinism and indeterminism, their applicability for classical and modern physics (and Reality writ large), the definition and existence of time, the relationship between causation and determinism, the finiteness of the universe, the impact of observation and measurement on systems, extra dimensions of Reality, what makes Reality, and likely several other questions. This is obviously a philosophical minefield, whose present Gordian-knot-style disentanglement is not going to be uncritically embraced by some readers. That said, I believe that it is logically sound and it is consistent both internally and vis-à-vis the evidence gathered in the first half of this work, which provides a novel perspective on these age-long questions. Furthermore, the three-pronged structure of the theorem itself can shed new light on the very same issues, if only by accounting for the contradictory answers that have been given to many of the above questions by different scholars, depending on the context of the problem and, likely, their metaphysical persuasions.

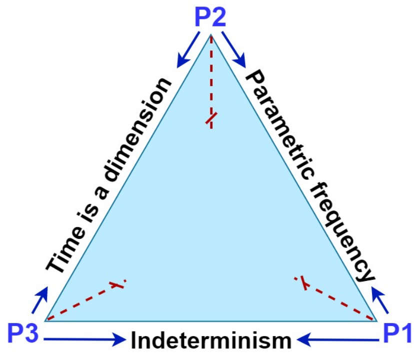

Figure 18: **The Triangle of Reality**. A graphical depiction of the three propositions of Theorem 1, formulated with inverse statements. Each one of the three propositions corresponds to a vertex, whereby the two adjacent sides to the vertex are true (marked with blue arrows) and the one opposite to it is false (marked with incomplete angle bisectors in dash red).

P2. Beginning with P2, we assume determinism, so the past and the future are predictable given the present knowledge, using physical laws, at arbitrary precision and to an arbitrary extent in time (§ A.1). This prediction should be expressible as a time function or time series of some measurable quantity, which can then be transformed using the Fourier integral to yield a corresponding frequency representation. The frequency representation is completely determined by the time function, which means that frequency on its own is not independent, and thus not a dimension.

There are two options here regarding time: it is either a parameter or a dimension. In the former case, the absolute value of the time parameter is not important, and the system dynamics is essentially time invariant and can be arbitrarily shifted to any point in the remote past or future with no measurable effect. Now, without loss of generality, the description of the system consists of both periodic and aperiodic components. Aperiodicity can be observed in both finite and infinite time extents, as long as it is modeled over the entire time domain (either through windowing or through a natural "windowing" of a finite universe age) that is required to eliminate phantom periodicity in the remote past and future (see Fig. 11). Aperiodicity of any kind, though, is characterized by a continuous spectrum with an infinite frequency bandwidth (according to the compact support paradox, §3.5.6). However, this contradicts the non-dimensionality of frequency, because parametric frequency cannot have infinite support and be defined over continuous arbitrary values if time is parametric too. This is so since parametric time in this context represents yardstick durations constructed from arbitrary periodicity measures that are abstracted from the time domain. The periodicity axis can only have a chance to map time if the continuous frequency domain includes zero frequency, which was introduced for convenience in the derivation of the Fourier transform limit. But, zero frequency is physically incoherent as long as time is parametric, because it encompasses all of time and cannot be arbitrarily shifted forward or backward, which makes it dimensional—a contradiction (§3.4.2). Therefore, no kind of aperiodicity can be obtained with parametric time.

Components of perfect periodicity can only occur with infinitely-long extent of time, or else there would be infinitesimal indicators of aperiodicity imposed on the time function measured—finite duration or finite damping (see further discussion in §A.2)—which contradicts periodicity.

We are thus left with only one option that would potentially violate P2: periodic dynamics that persists over infinite duration may appear to describe a deterministic Reality (a *deterministic* system) that has both time and frequency as parameters and not as dimensions. Now we are confronted with a different question: is there any physically realizable system—one that can be considered part of Reality—that is infinite in duration so it can admit perfect periodicity? In order to make this system realizable, it must be completely isolated, so no external forces whatsoever can interact with it and disrupt its perfect periodicity. This also applies to any kind of measurement that is required to ascertain whether the periodicity is perfect. The only truly isolated system that may count as such is the entire universe. But given that we are part of the same universe, we must interact somehow with this periodic system to learn about its present state in the first place, which was a precondition to enable determinism—it is necessary to peek at the system for "*one instant*" (§A.1). That moment of interaction, however—no matter how short and weak—is an aperiodic event, by definition. Then, our interaction constitutes a brief aperiodic disruption in the acquired time function and therefore contradicts the requirement that time is parametric, since the measurement introduces a unique moment that is no longer time invariant. Alternatively, we have to give up determinism that in this case refers to our knowledge that the isolated system is truly periodic, or to the availability of the Fourier spectrum, which would map to P1 and not to P2.

Therefore, determinism requires dimensional time and parametric frequency.

P1. P1 was essentially demonstrated through the proof of P2 above. Another way to look at it is to begin by assuming that time is not a dimension. This implies that the system being analyzed or observed can be described statistically as a stationary process, for which there is no difference in the choice of a reference time point, as there is no meaning to past and future—every point is as informative as the present and no absolute moment in time is special. This directly entails that the system is conservative and its dynamics has no beginning and will have no end. Thus, the notion of determinism—accurately predicting the past or future according to some physical laws—is incoherent. Frequency, if it has any meaning, relates to average periodicity patterns that are observable along corresponding time intervals—standalone parametric durations that are abstracted from time as a dimension that should have past and future. Thus, frequency describes the stationary physics and is not a free variable either, and hence not a dimension.

P3. Finally, in P3, we start from frequency being an independent dimension, which means that it can vary more or less independently with time. A fortiori, therefore, time is a dimension too. Inasmuch as the spectrum represents the physics of a given system, it is not constant in time, so it is only as informative as the information gathered in the present moment that defines it (i.e., over a finite time window). Therefore, unlike the P2 spectrum, it contains either limited or no information about the remote past and future, and therefore does not correspond to a Reality in which the remote past inevitably causes the present, which will cause the future. In other words, different pasts may have led to the present, which can in turn lead to different futures. Therefore, determinism does not apply here.

Hence, P1, P2, and P3 are mutually exclusive.

To complement this proof, we have to examine that the remaining five propositions that can be formulated using time, determinism, and frequency are never true (all propositions are summarized in the truth table of Table 2). Let us try to understand what each one of these statements entails and why it may be impossible:

~~P4~~. Time is a dimension; determinism; frequency is a dimension—Although time and frequency are two degrees of freedom of the system, its present behavior is completely determined by its past and its future is predetermined. Therefore, it is possible to obtain the frequency at all times from the predetermined time signal using the Fourier integral. But this means that the frequency is not independent of time and hence it is not a dimension—a contradiction.

~~P5~~. Time is not a dimension; no determinism; frequency is a dimension, *and*,

~~P6~~. Time is not a dimension; determinism; frequency is a dimension—For this proposition and the previous one to be true, it must be possible to continuously vary the frequency f in arbitrary steps δf and observe a respective change in time δt. But any change in frequency must only take place in space, if time itself is not a dimension. However, a spatial change in frequency would make time nonuniform across space, which would contradict its non-dimensionality[56]. Hence, these propositions are incoherent and the question of determinism is moot.

~~P7~~. Time is not a dimension; determinism; frequency is not a dimension—This was ruled out in the above proof of P2.

[56] An equivalent statement can be made based on Whitham's *wave conservation* formula, which relates the change in spatial frequency k in space and in time with $\frac{\partial k}{\partial t} + v_g \frac{\partial k}{\partial x} = 0$, where v_g is the group velocity in the medium (Whitham, 1999). The equivalent expression using the temporal frequency is given by $\frac{\partial \omega}{\partial t} + v_g \frac{\partial \omega}{\partial x} = 0$ (Weisser, 2021, pp. 57–59). This expression is completely generic, as it applies to any wave whose phase function can be written in the form of $\phi = kx - \omega t$, where ω and k are tied through the dispersion of the medium, so $\omega = \omega(k)$ and $k = k(\omega)$ (§3.1.3). If either k or ω change as a function of space or time, then one of the terms in the conservation equation is nonzero and must be balanced by the other. In particular, the second expression contains the derivative of the instantaneous frequency $\frac{\partial \omega}{\partial t}$, which is meaningful only if both ω and t are dimensional, and it may be nonzero only as long as $\frac{\partial \omega}{\partial x}$ is nonzero too, as is implied by propositions P5 and P6. Therefore, frequency cannot be dimensional without time being dimensional as well and we reach a contradiction.

#	$\bar{t}$	D	f	Availability
P1	**Yes**	No	No	**Yes**
P2	No	**Yes**	No	**Yes**
P3	No	No	**Yes**	**Yes**
~~P4~~	No	Yes	Yes	No
~~P5~~	Yes	No	Yes	No
~~P6~~	Yes	Yes	Yes	No
~~P7~~	Yes	Yes	No	No
~~P8~~	No	No	No	No

Table 2: Truth table describing the different available and unavailable combinations of non-dimensional time, determinism, and dimensional frequency. The following symbols are used: $\bar{t}$ – Time is not a dimension; D – Determinism; f – Frequency is a dimension. Only P1–P3 are possible, whereas P4–P8 are never true.

~~P8~~. Time is a dimension; no determinism; frequency is not a dimension—Regardless of the degree of knowledge about a given time signal, its respective frequency representation can never be fully recovered from it using the Fourier transform due to indeterminism (§A.1). For this to be true, it means that frequency reflects a degree of freedom that is independent of time, which contradicts its non-dimensionality. ■

Another way to break down the complex statement of Theorem 1 is to note the number of pathways that exist for time, frequency, and determinism to take place or not, enunciating Fig. 18:

- Time is not a dimension (1 pathway)
- Determinism (1 pathway)
- Frequency is a dimension (1 pathway)
- Time is a dimension (2 pathways)
- No determinism (2 pathways; *see §5.2)
- Frequency is not a dimension (2 pathways)

Theorem 1 was derived by way of deduction and elimination, by contrasting the different definitions of frequency and how they relate to time. We employed the Laplacian definition of determinism without challenging it, as it organically coincides with the underlying logic of wave physics, as well as with both *deterministic* and stochastic signal processing. However, different definitions of determinism (e.g., Earman, 1986) may lead to some refinement over how this concept was used in the proof. Still, it should not threaten the very interrelationship between time, frequency, and determinism as Theorem 1 has uncovered. While time and determinism are topics that have been often considered in the physics and philosophy literatures, the addition of frequency into this discussion is novel. The theorem itself, though, may be understood at different levels of abstraction, with more or less metaphysical baggage that was not originally purported at the outset of this exploration. Nonetheless, we shall make a few cautious strides and try to unpack a few of aspects of the theorem, before delving into concrete examples.

5.1 Choice of system: isolated, closed including losses, and open

There are at least three ways to understand Theorem 1. One way is to take it as an ontological statement, directly pertaining to the entire universe, as long as the three concepts of time, frequency, and determinism are employed as they are in present-day science. This understanding would have the universe fixed on either P1, P2, or P3. If it is the latter, then the frequency dimension is not merely a mathematical convenience, but rather a part of the physical Reality which is partially detected by our senses, although it is largely kept hidden from us. We shall not be dealing with this kind of interpretation directly, but only briefly comment on it in §5.2 and revisit it in §9.8.

The second way to understand the theorem and its three propositions is less rigid and allows for the propositions to apply in different physical situations, corresponding to the three abstract systems depicted in Fig. 13 and thus to three possible modes. P1 corresponds to the situation in Fig. 13 A, in which a system is truly isolated, so we have no access to it as such, as long as its boundary remains intact. Without being on the inside, the best we can do is to get average quantities that are time invariant, based on what we may a priori know that the system contains. This is another way to say that time is not a true dimension here, because it plays no role in the dynamics, for all we know. P2 is a closed system that includes the whole universe, as is illustrated in Fig. 13 C. It is fully conservative, but it is possible to define loss within the system. If this system corresponds to the whole universe, and it is known to be finite, then it already includes everything, by definition: it is impossible to include additional information (more forces, loss of energy). Hence, we have determinism, as the past dynamics causes the present and future. The inclusion of loss mechanisms that dissipate energy and information within the system gives meaning to time as a dimension as we intuitively understand it, as it gives rise to dynamics that can be clearly associated with cause and effect. If we breach the system boundaries (or if the universe is not isolated), we would be moving to P3 and the open system of Fig. 13 B. This mode remains in effect as long as the system and its environment retain their identities. If they are merged, or taken as a whole either analytically or in observation, then we return to P2 (Fig. 13 C).

A third way to understand Theorem 1 is epistemological and concerns avenues to acquire knowledge about the world through observation, which pertains both to scientific measurements and to sensation and perception, and can be used to construct a model of Reality. The three propositions here can be seen as different methods to approach Reality—each of which requires different degrees of data availability, assumptions, computation, and, potentially, previous information. In this vein, P1 relates to the minimum amount of assumptions and coarsest data acquisition—obtaining a time series about which we may know little or nothing, but assume that if it varies, then it is around a mean. This allows us to obtain a baseline statistics of the observed object. We can increase the fidelity of our measurement by interacting with the system in addition to passively observing it. This likely includes transfer of energy that, if done right, can be accounted for and might not affect the measurement. Here, fluctuations around the mean are taken as relevant information that may have to be modeled and interacted with. We can take whatever knowledge we have about all this and relate it to observed generators in the system, which mechanistically cause it to behave in the way we observe that makes Reality. This is P2. We can also use this information in a P3 mode, which ideally requires us to specify a time window and bandwidth, and then enables us to obtain a much closer handle on any temporal variations that deviate from the stationarity assumption of P1. Both P1 and P3 may be used to approach a P2 conception of reality, which we would like to match with Reality—a 4D world where causation plays a role in instantiating predictability, which enables us to exist with relative confidence about things (see §9.7). Sometimes we can directly approach P2 based on patterns we obtain in real time that allow us to induce a behavior regarding the history and future of the signal and its origin. But this may only be done conditionally, given that our knowledge of the remote past and future is either limited or altogether nonexistent, which means that any predictions about them may be prone to error (or noise).

The examples in §6 will illustrate several aspects of these interpretations.

5.2 Determinism

Determinism appears in two varieties here—explicit and implicit. Explicitly, within the epistemological interpretation of P2, there is the low-level Laplacian determinism (see § A.1) that follows from the interdependence of frequency and time. It encompasses every event, movement, and bit of information to have ever existed to be related to produce the time signal or observation at hand. There are no real inputs nor outputs to such an isolated system we call the universe, because there is nothing external to it, by its very definition. This inevitably leads to a predetermined future, since the information about it is already contained in the system's past.

Implicitly, another form of determinism emerges from the theorem, if it is interpreted strictly ontologically in a manner that entails an unyielding maintenance of the system boundaries, or relationships with other systems that are internal or external to it. If it is an isolated system of the form of P2, it is akin to a mini-universe of a finite extent. As long as its boundaries are maintained, implicit determinism is maintained, in the sense that the proposition associated with it becomes an immutable mode of being. In other words, this sort of determinism implies an adherence to a fixed mode rather than moving between the three modes. In this sense, an isolated system that corresponds to P2 and remains as P2 is doubly deterministic. However, the definition of the system can be tinkered with by letting its boundaries vary irregularly, which can cause a change of mode (i.e., corresponding to P1, P2, or P3), which is an alternative avenue to relax determinism.

5.3 Indeterminism

Indeterminism appears here twice—both in P1 and in P3. Yet, as was suggested throughout the preceding analysis, the nature of the two is fundamentally different. In P1, indeterminism is associated with probability and statistics. In this mode, all quantities are distributed around a mean and can be characterized using correlations and moments. The predictability thereof is constrained by the sharpness of the distribution that describes it. Given the law of large numbers, after a sufficient number of records, we would obtain the relevant expected values of the system with diminishing uncertainty. It does not mean that we should be able to predict the next sample with precision, though, as the precision describes the ensemble and not its individual members. That said, we can still form predictions about temporally proximate behavior of individual samples based on the autocorrelation of the ensemble. However, it is a probabilistic and not a causal measure, so the predictions can never be certain for autocorrelation that is less than unity, $|R_{xx}| < 1$. In contrast, When we talk about the ensemble behavior in time (if observed from outside), we know how it is going to be in the remote future and past using induction—there should be no great surprises here, as long as the probability distribution we have is correct.

The situation is markedly different in P3. Here, one "rides" or "surfs" the wave in real time, using the instantaneous quantities and measures and the general knowledge about the associated bandwidth and its center frequency (or set of carriers, if it is a broadband signal). This affords arbitrarily precise knowledge of the immediate variations of the signal—constrained by analytical and technical limitations—but provides no information whatsoever about the remote past and future. Therefore, it is an antonymous kind of indeterminism to the one experienced in P1.

This form of near-determinism is reminiscent of the concept of the *Lyapunov time* from chaos theory and nonlinear dynamics (e.g., Gaspard, 1998). Unstable chaotic systems are known to be sensitive to small changes in initial conditions, which are amplified to a gradually increasing error the longer time has elapsed from the initial time. The Lyapunov time is the characteristic time beyond which the error becomes larger than the precision of the measurement, so the system behavior may no longer be predictable from its *deterministic* dynamic equations, while only general statistical descriptions can be validly made.

A related idea regarding different degrees of indeterminism is found in Popper (1965 / 1994, p. 220), who was searching for something that is neither hard determinism, nor hard indeterminism (Emphasis in the original): "*While physical determinism demands complete and infinitely precise physical predetermination and the absence of* any *exception whatever, physical indeterminism asserts no more than that determinism is false, and that there are at least some exceptions, here or there, to precise predetermination.*"

Taking the notion of precision as key, we may amalgamate all of the above, by underscoring that P1 describes the temporal horizons, whereas P3 describes the temporal vicinity. It stands to reason that information gathered from both provides a picture of Reality that is closer to deterministic, and hence, to P2 (see §9.7). Any deviation from precision in prediction of near or remote events should result in uncertainty, error, distortion, or noise—variably depending on the context.

6 Epistemic examples

A few examples that illustrate how Theorem 1 may be applied and interpreted are sketched below. Initially, they depict relatively mundane situations and systems, in order to demonstrate the reach of the theorem. In all cases it is shown how a combination of P1, P2, and P3 perspectives leads to effective solution of problems within these situations, often in complementary ways.

6.1 Example I: Traffic flow

The first example is deliberately basic, in order to keep it relatively relatable. It serves to distinguish the three modes and their applicability in different circumstances, although they clearly refer to the same Reality. Note that this example bears no relation to standard analyses of traffic flow problems (e.g., Lighthill and Whitham, 1955).

Let us consider a one-way single-lane busy road that is congested with heavy traffic during peak hours, which drops to a minimum at around midday and at night. Additionally, the traffic depends on the day of the week, with a drop on the weekend, and seasonally, with heavier but slower traffic during the winter. We would like to get a handle on the traffic dynamics and use the data for different actions. We set a detector that can precisely record the exact time and date in which the front bumper of every vehicle passes a certain point x_0 on the road. The detector is also able to record the rear bumper of the cars, associate them with their respective front and infer their instantaneous speed at point x_0. We would like to estimate the variation of the air pollution—a separately measured set of time-series data—around the road as a function of car density. To greatly simplify the problem, it is given that there are no other roads or sources of pollution that may contribute to the pollution level in that area.

This problem is readily solved statistically in P1. We can sample all the cars over a certain period—say, one year—to compute the average frequency of cars on the road per unit time[57]. We would then obtain a grand average over all days and hours. On top of that, we can have a "power spectral density" statistics of cars, whose peaks reflect the different periods of high and low congestion during the day, week, and month. Each peak will have a "bandwidth" that represents the spread of observed frequencies that contribute to the average around that peak. From these numbers, it should be fairly straightforward to derive a correlation model between the car frequency on the road and the level of specific pollutant. The same would be true for traffic noise level estimation, the tendency for traffic jams to occur, for estimating of the average number of accidents on the road, or for computing the wear of the asphalt and the tires on the road in comparison with other roads. When extrapolated to all of time, these measurements are assumed stationary in the sense that the absolute time point should make no difference for the data observed and extracted from this system. Some refinement can be achieved by independently factoring contributions from daily, weekly, monthly, and seasonal statistics that are allowed to modulate the total average car density, in what is called *cyclostationary* analysis.

This frequency data, however, would be virtually useless if one wants to know at which specific moment it is safe to cross the road (assuming drivers do not generally slow down when they see a pedestrian crossing the road). Except for being able to tell when the road is busier, crossing the road based on average rather than instantaneous frequencies of cars on the road would be, by and large, suicidal. Therefore, in that case, one works in P3 by picking a particular temporal window that is sufficiently long for oneself to detect a coming car and cross to the other side in one piece. When the car density is higher, this time window is, on average, going to be shorter. Whereas, during low traffic hours, the time window may be almost as long as desired. The crossing individual may be able to instantaneously estimate the oncoming car velocity in order to match their own speed necessary to cross the road. The car velocity itself cannot tell much about the car density, but if it is integrated over a certain duration, an instantaneous car density function may be obtained.

Finally, we may want to identify the exact time course of a particular car, whose driver leaves her home every morning at sunrise, seven days a week, and arrives to the road exactly five minutes later. Here we resort to P2. We adjust our detector to record only that one car in- and out-times at x_0 and, just like in the first (statistical) case, obtain a spectrum with a clear spectral line corresponding to one drive per day. But the line has some "bandwidth", because of the seasonal oscillation in the precise hour of sunrise, as well as some random car traffic around the time when the driver passes the detector, or the driver's regularity—all of which result in some arrival time uncertainty. If we precisely measure the spectrum over, say, a month, and given a hypothetical exact periodicity of the years, this spectrum will enable us to compute the exact time of day in which the car passes the detector, subject to the random fluctuations in traffic around that time and to the driver's state. In any case, this spectrum is going to look nothing like either one of the other two spectra—it is irrelevant to crossing the road at noon, just as it tells us nothing about the average (generic) car frequency on the road throughout the day. However, if we repeat the measurement for each individual car and average the result, we would get the power spectrum of P1, only with the level of detail that may allow us to also cross the road safely, if only in hindsight. Therefore, at the limit of full details of all car data, we get a deterministic picture of Reality, from which we can derive both the individual car data and the ensemble averages.

6.2 Example II: Measurement of target sound pressure level with external disturbance

Let us consider another relatively basic example, this time from bioacoustics, although suitable analogies can be drawn within other fields. Suppose we are interested in the cicada mating song, which is produced within male choruses of genus *Magicada* every 13 or 17 years, periodically (Williams and

[57] The frequency we use here converges with the definition of relative frequency as is employed in frequentist statistics (see Footnote 38). However, it is trivial to convert the measured time series to physical frequencies, if every car detected triggers a pulse of voltage or light, for example, and is then registered by an observer, in a more similar form to other oscillatory event detection. Clearly, though, there is no limit on how sophisticated the conversion may be between the car kinematics and dynamics to an oscillation resembling a more readily workable signal or a wave.

Simon, 1995). Specifically, we would like to examine what the very first song of the season sounds like, and what its sound pressure level and spectrum are. Let us assume that we have a guesstimate of when and where the first song is going to begin within the cicada habitat, and we intend to record it using a calibrated (reference) microphone.

We can model this acoustic setup according to the (linear) wave equation, which incorporates the positions of one source (the cicada) and one receiver (the microphone) in the inhomogeneous term, with optional boundary conditions to account for the effects of the soil and tree reflections. We may be able to precisely estimate the level of the cicada's song at its own position using results from this model that is firmly embedded in a P2 mode of Reality. However, suppose also that at the moment of the recording (observation, measurement), a loud airplane flies over the microphone and its broadband sound corrupts the song recording. In that case, we could consider dropping the recording, since the airplane noise contaminates the clean recording of the cicada, leading to poor signal-to-noise ratio. We would never consider the wave equation to be wrong, but rather the system that it describes to be incongruent with the system in actuality at the moment of recording, which contained two sources rather than one. Obviously, we could have used the very same wave equation to model both the moving airplane and the cicada—including them both in a single closed system—and it would be completely correct (neglecting nonlinearities associated with the airplane dynamics and interaction with the atmosphere). But in all likelihood, we would have no motivation to invest so much effort in this difficult problem.

Instead, we may try to repeat the measurement when the external disturbance is gone and assume a certain relationship between subsequent songs and the first one whose recording was corrupted. For example, we can look for another cicada population, or wait another 17 years for the same population, and try to record its first song. This will again lead us to a P2 mode. Or instead, we can switch to a P1, statistical form of observation: we can record other isolate songs of the same population (before the full chorus begins), average them, and assume that the average is identical to the first one, irrespective of when they appeared and which cicada produced them. Or, we can estimate the number of cicadas during the chorus song, record it, and make inferences about the individual song by dividing the long-term power spectrum by the number of cicadas. Or, we can sample different airplane sounds and subtract their average power spectrum from the corrupt recording, perhaps even using a sophisticated machine-learning or artificial-intelligence algorithm tailored for this special purpose. And so on and so forth. All variations induce some form of time invariance and deviation from a certain mean song pattern. Depending on the specific assumption used as the basis for each procedure, it almost unavoidably incurs information loss of the original recording that may have been singular and cannot be exactly reproduced.

Alternatively, the spectrum of the airplane noise may turn out to be relatively low frequency, whereas the song may be relatively high frequency. In that fortuitous case, we may prefer to high-pass filter the cicada's call from the corrupt recording, containing both the airplane and the cicada sounds (in complex cases, filtering can become proportionately complex; Wang, 2005). If we select an appropriate cutoff frequency, the error would be minimized, and most of the energy contained in the song will be accounted for and yield a reasonably precise estimation of its level, as though it was made in an acoustically isolated, airplane-free system. For all intents and purposes, the entire solution to this problem is worked out in five dimensions and, arguably, in a P3-like method, which was matched for the particular, instantaneous bandwidth and moment in time. We circumvented the rigid application of the closed-system wave equation without sacrificing its validity, by the addition of noise and then removing it, however possible, through filtering. We will report the result along with the effective bandwidth that was used to obtain it, which indicates the band-limitation applied to the cicada's song spectrum.

In practice, information gathered from all methods can be combined to reconstruct an image of this highly localized Reality in the form of one song that appeared at one very particular moment in time, to never repeat. All methods are at least partially correct and complement one another, but they all rely on different assumptions that incur some uncertainty on the fidelity of our knowledge about that moment. Inevitably, a certain level of noise must be acceptable, which should not impact the validity of the model being tested, or the equation that governs it, so the notion that the system under test is indeed acoustically isolated may be retained. The noise source could have been part of the governing differential equation, if we only set the boundary of the problem differently and managed to get it to correspond to the acoustic Reality in the field.

Replacing the cicada, airplane, and particular choice of filtering applied to the observation, this otherwise simple problem description can fit numerous other problems without loss of generality in which the noise level cannot be ignored or controlled, but the noise source itself is of no direct interest in its own right. In most cases, though, we can semantically avoid working with an extra dimension, simply by calling it "noise" and accepting it as an inevitable discrepancy between the sought-after and the obtained solutions, so that a deterministic knowledge of the observed event can only be obtained within a certain margin or error / fidelity. This implies that we sacrifice a degree of determinism that is afforded by differential equations and Fourier analysis and convert it to noise. Reality is no less deterministic as a result, but our access to this determinism is constrained by different forms of noise[58].

6.3 Example III: Radio communication

Communication engineering encompasses several domain-general methods[59] on how to optimally transmit and receive arbitrary messages between two points in space over a noisy channel. It lends itself as a quintessential test case for Theorem 1, because these methods involve a regular theoretical and practical employment of critical information gathered using all three modes: P1, P2, and P3. Both transmission and reception unfold over time and both are susceptible to noise from multiple sources picked up along every stage of the communication chain. In this context, there is a sharp distinction between the "signal", which is the desirable message to be communicated—what was originally sent—and "noise", which is the undesirable disturbance that is picked up along the way and is processed by the same circuitry as the signal and is present at the output. While random noise is an inevitability and cannot be altogether eliminated, a well-designed communication system aims to minimize it and deliver an undistorted replica of the original signal (i.e., an output signal that is identical to the input, with the exception of a linear amplification factor). An assortment of methods from mathematics, signal processing, stochastic processes, information theory, physics, and electronic engineering is commonly deployed in the design and modeling of communication systems to realize this goal.

In this example we focus on *bandpass communication*, which is the standard in long-distance communication, and attempt to give a bird's eye view of the generic logic that guides its realization (e.g., Couch II, 2013). The input here is an arbitrary *baseband signal*, which occupies a bandwidth between zero and some cutoff frequency, $B > 0$, that is modulated on a carrier frequency of a much higher frequency, $f_c \gg B$. The process of modulation involves a nonlinear transformation of one of the carrier parameters (amplitude, phase, or frequency) using the baseband signal for the transmission. An inverse operation is applied for the reception, whereupon the baseband signal is recovered (or reconstructed). *Baseband communication*, in which no modulation is applied, tends to be impractical over large distances and is not discussed here.

At every point along the receiver processing chain we can query what we know about the signal, which is otherwise physically identical: it is what can be measured in an observed portion of space, typically reduced to a single point, in which a continuously varying field energy, usually electromagnetic, is being detected. Information about every aspect of the communication system is expressed using one of the modes, as will be illustrated below. In all cases, the output from the detector / receiver is a scalar time function—predominantly voltage—that can be either further processed or in some cases sent directly to an output device (a loudspeaker, a video display, an actuator, etc.).

In general, the receiver "does not know" what signal it is going to receive, so the only way to design for it is to statistically specify it beforehand. This is done using very general assumptions about the probability distributions of the received noise, and combining it, at minimum, with the detection rules for the specific modulation, the modulation signal bandwidth, and other elementary parameters of the communication channel. All these are modeled as combinations of *deterministic* and random processes, which result in time-invariant statistics that can guide the system design[60].

[58] In open systems, different sources of modulation normally seem as noise if measured using standard Fourier spectrum (i.e., P2) or power spectrum (P1), but their periodic origin can be uncovered using instantaneous frequency methods, à la P3 (Rowland Adams et al., 2023).

[59] By "domain-general", it is implied that the bulk of the analysis done in communication theory is purely mathematical. The practical constraints imposed by things such as physical wave propagation, electronic equipment, computational and signal processing limitations are not generally part of the initial characterization of the communication process.

[60] The assumptions of time-invariance and stationarity do not hold for complex communication scenarios, when at least one of the transmitter / receiver pair is in movement, or there are other variable physical conditions that restrict the stationarity. In these cases, some *deterministic* components inevitably enter the channel modeling (e.g., Han et al., 2022). Even then, it is a standard practice to identify the conditions and time intervals during which the channel can be treated as effectively stationary, so that a suitable statistics can be derived based on it (e.g., Steinbauer et al., 2001; Ghazal et al., 2017; Han et al., 2022; Schwartz et al., 1995, pp. 343–415).

Here, the carrier is typically taken to be a stationary *deterministic* process that is mixed with stationary random process—noise. Similarly, the modulation signal is typically taken as a *deterministic* process (it may be also mixed with *indeterministic* noise). Using knowledge of the modulation and demodulation transformations, it is then possible to predict the best signal-to-noise ratio that is attainable for a given type of communication, which may depend on frequency. In digital communication these considerations can be used to predict the *bit-error rate* for a particular modulation method as a function of noise level. Various sensitivities and caveats can be flagged and alternative methods can be compared, also based on physical feasibility. Thus, this characterization of the communication system is fully found within the realm of P1.

Another critical characterization step of the transmission is most informatively realized in P2. Here, the Fourier spectrum of the modulated transmission itself is directly calculable, so that the effect of the modulation operation on the spectrum around the carrier frequency can be scrutinized. The information obtained here is used in different ways. For example, any type of modulation gives rise to *sidebands* in the spectral vicinity of the carrier. In selecting the channel and modulation types, it is conventionally required to avoid overlap between the sidebands and any simultaneous adjacent transmission (either its carrier or sidebands), so that the signals will not interfere and cause severe degradation in the quality of the attainable demodulated signal[61]. Another use of the *deterministic* spectrum is to design alternative modulation methods that make better use of the energy that is used for the transmission, or in the bandwidth it occupies. For instance, this can be done by eliminating one of the sidebands, or by using quadrature modulation using the same bandwidth[62].

When it comes to the electronic circuit analysis, a combination of methods is typically employed to find the voltages in the different circuit nodes. Although the response may be based in some cases on differential equations such as Eq. 24, it is normally reduced to algebraic equations instead, which result in a *deterministic*, time-domain prediction of the voltages (and currents) for a given input voltage and frequency (P2). When the circuit is analyzed with respect to its response to random noise, then P1 methods are invoked again, where the power transfer function is used for both signal and noise.

Finally, the signal that is being demodulated is represented in the time domain. It may seem deterministic at the present moment, but there is no knowing how it is going to be in the future or how it was in the past, so the only thing that may be known about it with any degree of certainty is its estimated long term average power spectrum, or level distribution. But (for the person) at the receiver's end, it is the instantaneous, nonstationary aspects of the signal that are of value rather than its average properties. So the entire analysis of the modulation and demodulation processes is done while preserving the temporal structure of the signal—any deviation from it would lead to distortion. Specifically, when dealing with frequency modulation, it would be self-defeating to have signal representations that exchange the instantaneous frequency with series of infinitely long parametric frequencies (see § 3.4.7). This is true both in cases where the signal is detected using *coherent* methods—by employing a local oscillator that synchronizes to the phase of the carrier (see § 3.5.9)—or *noncoherent* methods, in which the modulation frequency is extracted without synchronization (for example by directly converting it to amplitude modulation). Therefore, in the most general sense, this part of the analysis requires a P3 perspective[63].

All in all, an epistemological interpretation of Theorem 1 appears to be reflected in the communication system design. Each one of the three modes, P1, P2, and P3 is used to illuminate another aspect of the system and the signal that either cannot be effectively obtained through another mode, or it may be downright impossible. Regardless of the mode chosen, the physical signal and the system remain the same all along. The choice of mode becomes either a mathematical convenience or a necessity, without which some problems may be otherwise intractable. This is strongly reminiscent of Slepian's resolution of the bandwidth paradox, where he made a distinction between the mathematical tools we use to understand, characterize, and measure signals and the actual physics in Reality that is indifferent to these tools (Slepian, 1976; see Footnote 28).

[61] Once again, there are exceptions to the rule with some advanced communication systems. *Spread spectrum* communication systems may incorporate digital modulation techniques with suitable coding of the messages that is designed to work despite overlap with other channels (e.g., Proakis and Salehi, 2014, pp. 825–869).

[62] *Quadrature modulation* is the independent modulation of the two amplitudes $x(t)$ and $y(t)$ of the orthogonal components of the signal $s(t)$ when it is expressed as $s(t) = x(t)\cos\omega_c t - y(t)\sin\omega_c t$.

[63] It should be noted that, mathematically speaking, even strict amplitude modulation that is not continuous still gives rise to some frequency modulation and vice versa (Picinbono, 1997). The two types of modulation cannot be completely disentangled in the reciprocal domain, as it is the quadrature amplitudes that are statistically independent, rather than the amplitude and phase. Nevertheless, in practical applications the modulation types are treated as independent.

An epistemological interpretation of the theorem in the context of the communication example leaves the question of an ontological interpretation undecided. However, we can consider a few additional points, which circle back to our own everyday experience and may have an ontological bent. A radio transmission unfolds over time. To the listener, the spectrum and time signal, or the statistical description of a song that is playing in the radio now, does not provide any predictive power for the contents and sound of the morning news show that is going to play next Monday. Also, that song provides no information whatsoever about another song that is playing simultaneously on another channel, is transmitted by the same equipment, and may be received by the very same radio receiver without moving any parts, only through minute changes in the configuration of some bits of software and / or hardware. Physically, the carrier waves of the different channels are in superposition throughout the medium and many of the communication system parts, but the various modulation, demodulation, and filtering operations ultimately determine what is going to be heard in time and in frequency. Reinstating the a fortiori argument from §5, unless we insist that the time signal has long been predetermined and its future is fully known, considering only time but not frequency as an independent dimension of Reality would be logically inconsistent in the context of radio communication.

Another aspect of the theorem that is not clearly reflected in the communication engineering problem is whether the three modes are indeed mutually exclusive, given that they are used side-by-side to model Reality. Short of going through specific models and derivations one equation at a time, it would be difficult to demonstrate how at each step only one mode is used at a time, despite several hybrid steps in which *deterministic* and stochastic processes are used simultaneously. For example, P1 relates to a statistical approach, which averages over time to the point that time does not exist as an independent dimension. However, in nonstationary cases, it is common to divide time to discrete units (e.g., frames, as was discussed in §3.5.6), where the stationarity assumption can locally hold, or where any of the other two modes can be applied without violating the theorem or any of the problem assumptions. This somewhat confusing nesting of local and general realities is not inconsistent in its own right. Rather, it indicates that the theorem may apply on wildly different scales, which may include anything from the entire universe, to a microscopic region within it. For this statement to be fully justified, though, we would have to explore whether the theorem applies to quantum problems, which relate to the smallest-scale features of Reality and constitute the most reductive point of view about it (see §7 and §8).

6.4 Example IV: Reaction time to ambiguous words

Our final epistemic example is taken from the field of psycholinguistics and may be considered somewhat of an odd choice to bundle with the previous technical examples, as well as overly associative. Nevertheless, it is included here both to emphasize the generality of the theorem, as well as provide a suggestive link to other situations in which mental processes are pertinent.

The general topic relates to the putative process in the human brain called *lexical access*, in which a word stored in the "*mental lexicon*" is matched by the brain to a word encountered in perceived (heard) running speech. The research question of interest is what happens when the brain encounters a word that has several different meanings that share an identical pronunciation (a *homonym*). A reasonable hypothesis is that ambiguities slow down lexical processing, because the brain must select the most congruent out of several meanings. A more refined question is whether the decision is facilitated when a context is available, which can indicate the correct meaning. This question arose some interest back in the early 1980s beyond the confines of the psycholinguistics field (notably in Fodor, 1983), because the influence of context seemed to be indicative of whether the brain processes language in a *bottom–up*, modular way—i.e., sequentially, from the auditory / linguistic input to the cognitive output, where each processing module in the brain is autonomous. Alternatively, the brain also exhibits *top–down* processing—feedback from advanced-stage processing that weights the early perception-related processing for incoming words—that can arguably make disambiguation more efficient, but may require a higher degree of neurophysiological sophistication.

Several simple, opposing processing models had been put forth and each had seemed to successfully account for different sets of observations with homonyms, while contradicting the results from others, suggesting that the actual brain process may be more involved than any of the models imply. However, no clear-cut resolution that resulted in a universally accepted model has been reached. The discussion and controversy seem to have largely fizzled out by the early 2000s, as the focus shifted to brain imaging studies that can reveal the specific pathway activation during specific cognitive tasks, including linguistic ones. It is not the intention here to introduce the intricacies of this topic in any depth, but rather to highlight its three most influential models and analyze them using the novel perspective gained from theorem 1. For a recent review of the topic see Rodd (2018).

The brain is exposed to spoken language sequentially, which means that the meaning inferred from it also unfolds over time, as more information is being communicated to the listener. The semantics of spoken language—the meanings that individual words carry—is used to infer the overall meaning of the sentence, in what is technically referred to as *integration*. When the semantics is ambiguous, integration can be made more complicated. Ambiguity of meaning can be manifested in many ways, but we focus only on ambiguity that is potentially present due to multiple meanings of words. The dependent variable observed is the *reaction time* of listeners—test subjects who complete a certain linguistic task. The underlying assumption is that the reaction time is dependent on the time that is required for the brain to process the linguistic input in order to correctly execute the task. In general, shorter reaction times imply simpler processing, whereas longer reaction times may indicate ambiguity-related delay.

Three classical models have been put forth to account for homonym processing in different tasks (summarized in Fig. 19). Roughly speaking, we argue that the three correspond to P1, P2, and P3 and that the discrepancy between them reflects the mutually exclusive nature of the three modes.

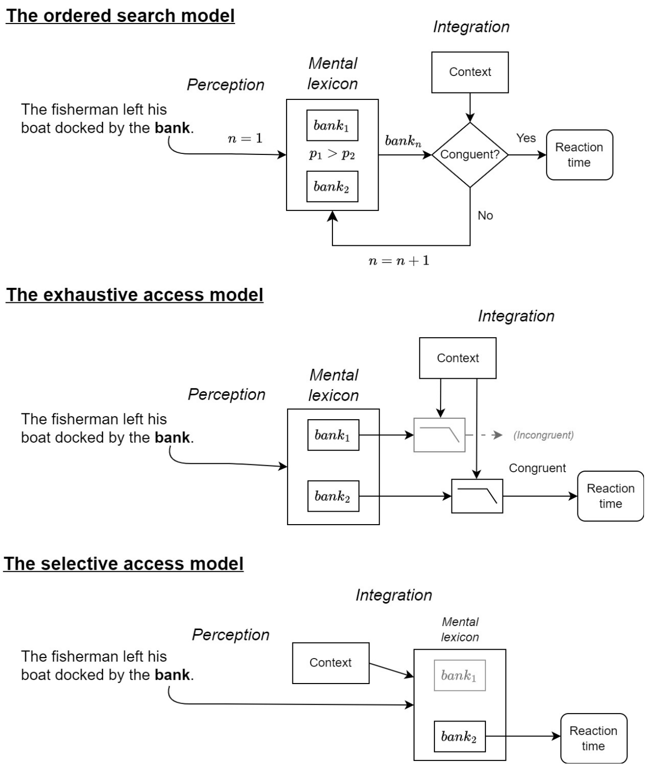

Figure 19: Three cartoon models of the mental process leading to the correct understanding of words with multiple meanings while listening to speech. In all three models, a hypothetical listener perceives the spoken words and responds to a task, which is measured by the reaction-time proxy variable. The words are represented in the brain within a "mental lexicon" that contains the different meanings. Only the sentence context can disambiguate the sentence and select the congruent meaning. In this case, the congruent meaning of the word *bank* is the less frequent one, designated with lower probability $p_2 < p_1$ and denoted with $bank_2$. In all cases, we tacitly assume that the context is formed over time from previous meaning integration. **Top**: The ordered search model goes by relative frequency of the meaning in the language. It first attempts to match it and moves to the less frequent (less probable) meaning only if the former meaning was incongruent with the context. **Middle**: The exhaustive access model excites all available meanings for the word and context is used to test all of them for congruence, whereby only the congruent meaning makes it through. **Bottom**: The selective access model uses the context to excite only the congruent meaning from the lexicon.

The *ordered search model* (Fig. 19, top) holds that the processing time of homonyms relates to the word meaning frequency (prevalence) in the language (Hogaboam and Perfetti, 1975). This model assumes that the lexicon attaches a certain probability to every word meaning stored in it and preferentially selects the most frequent one, unless it does not match the context. For example, the

English word "*bank*" referring to the financial institution is used more frequently than "*bank*" that refers to a shore of the river (Rice et al., 2019). Therefore, in a sentence such as "*The man walked to the bank.*," the former interpretation will be preferred according to this model. However, if the two meanings have the same frequency, then both will be accessed and context will have to be used to disambiguate them. This is a P1 kind of model, since any information about the frequency of word meanings is taken to be stationary at the time of testing—it is abstracted from the present and is assumed universally applicable. It is also not deterministic, but only heuristic, because there is no guarantee that the selection according to the highest-frequency word meaning is going to hold, until the interpretation is corroborated with the appropriate context. While it is unspecified here how the context itself may be generated, it is understood to be a separate process that is independent to the word meaning disambiguation.

According to the *exhaustive access model* (Fig. 19, middle), all word meanings are simultaneously accessed, independently of their frequency, and only later is the correct meaning selected using the context, whereas the irrelevant meanings are discarded (Conrad, 1974; Swinney, 1979). This model corresponds to the logic of P2 that is applied in the time domain and strictly follows the available information in real time, without assuming anything about the context or frequency. We can think of the ambiguous word as an input signal—a stimulus—that excites the system (the lexicon), whose output are certain meanings, which decay more or less rapidly according to the various excitations to the system. Figuratively, this model seems to reflect the impulse-response technique from dynamical system analysis, signal processing, and wave phenomena (where it is better known as *Green's function*), where it is possible to uncover all the modes of vibration of the system (see §3.1.2) by exciting it with a very short pulse (mathematically, a delta function). Information from impulse-response measurements reveals the complete dynamics of the system (if linear). Specifically, it enables us to predict which modes are going to resonate with certain input signal frequencies. Also by analogy, this process generates some transient uncertainty (noise) due to the associated uncertainty in the immediate time frame after the ambiguous words are encountered, but as the irrelevant meanings decay, it leads to a deterministic decision, where the uncertainty no longer plays a role (Onifer and Swinney, 1981).

Finally, the *selective access model* (Fig. 19, bottom) maintains that if a disambiguating context is available at the time when the homonym appears, then it is used to preselect the correct meaning (Schvaneveldt et al., 1976). Arguably, this is a P3 kind of model, because it is locally concentrated on whatever available information there is to minimize the difficulty associated with integration. Context, while difficult to quantify, can be thought of as providing some guidance for where to search for meaning, both in time and in frequency. Context forms and develops sequentially, so time is a dimension, and the search time window is therefore distinguished. The disambiguating context can be used to bias otherwise secondary meanings due to their low frequency, which entails a form of filtering (although not necessarily in a dimensional sense—it can be more simply thought of as a parametric operation in a non-physical meaning space). As long as the context inferred by the brain is correct, the result may be the most processing-time efficient of the three models, but it should require a feedback operation in processing. However, if it is incorrect, then the mistake may be more costly, as a correction has to be produced using another method. Indeterminism here relates to the fact that the processing implied by the model is local—it applies for a specific sentence and context and may not induce predictability to other sentences in the current or other conversations.

These three models are at odds with one another. There has been supporting evidence for all three and a number of *hybrid* models were proposed, which feature aspects from all three basic models by carefully tracking the difference in meaning frequency of words used, as well as the contextual information strength and quality during the listening task. Whatever information garnered by this kind of processing model may not be readily classified into one of the three modes. Nevertheless, the three basic models and their (perhaps strictly metaphorical) mapping to the three modes can be used to highlight two things about this example, which may be further generalized and contrasted with previous examples. First, the three modes (models) all refer to the same Reality, but harness different assumptions and methods to correctly observe it. Second, an approximate image of Reality may be constructed by combining information gathered by all three modes. This may refer to the information from these modes being integrated in series or in parallel, perhaps at spatially distinct loci. It begs the question—if elements of the three modes can be combined as they are in various advanced disambiguation models—are they indeed mutually exclusive? For example, suppose the internal knowledge about the meaning frequencies is indeed stationary, but its output can be biased (filtered) by certain contexts. In a sense, such a hybrid procedure breaks down the problem into smaller sub-problems, each of which is solved within a single mode. Regardless of how the process unfolds, the lexical decision must end deterministically, though, which calls for a P2 end point[64]. Therefore, reality appears to be constructed through the complementary information gathered by the different methods that are applied at different time scales. This makes the mutual exclusivity more subtle, if at all appropriate.

What we cannot know from this analysis is whether the three models and modes are fundamental to how the brain processes external stimuli in general, or rather how we, as conscious observers or scientists, conceptualize the brain doing so using language. There are indications, however, that point to the former. See further discussion in §9.10.

6.5 Mode transition

Theorem 1 states that the three modes of observing Reality are mutually exclusive, but does not say anything about the transition between the modes. Since the three modes ultimately relate to boundary drawing of a system by an observer—a boundary that may or may not have a physical manifestation—the transition necessarily relates to a (deliberate, imposed, or haphazard) change of boundary between or within systems. Therefore, one direction of exploring the significance of the theorem is to consider effects that potentially take place in and around mode transitions, where observational discontinuities or logical inconsistencies may arise. Six transition types are possible: P1 to P2, P1 to P3, P2 to P1, P2 to P3, P3 to P1, and P3 to P2. Three of them are considered below in the context of quantum mechanics.

As with the very propositions of Theorem 1, mode transitions may be understood epistemologically and / or ontologically. Invoking an epistemological point of view, a transition between modes relates only to how we get to know something about Reality through different mathematical operations and methods. So, for example, there is generally some information loss between P2 and P1, which means that our knowledge of a certain process may suffer as a result. Alternatively, there may be an ontological interpretation, which relates to how things are in Reality and whether they can be reliably mapped to one of the modes, but not to another.

In the next example (§7) the line between these two interpretations is further blurred as we apply the theorem to the measurement problem in quantum mechanics. The final example, of quantum nonlocality (§8), hypothesizes an ontological validity to the theorem and to the existence of frequency as a dimension, in order to see if it can account for the nonlocality that is the hallmark of certain problems in theoretical and experimental quantum mechanics. The two examples also serve the purpose of exploring how the theorem applies to simple systems, different from the above examples that all featured multilayered epistemology of engineered systems, which made it difficult to ascertain whether indeed no two modes ever occur simultaneously.

7 Example V: The measurement problem of quantum mechanics

7.1 Background

Quantum mechanics is the branch of physics that deals with molecular, atomic, and subatomic systems—the smallest physical systems in existence, as we understand the universe today. In its century-long existence, it has been as successful as it has been enigmatic and controversial, and as such it has occupied a central place in the imagination of physicists and enthusiasts alike. We invoke quantum mechanics for two convergent reasons. First, we would like to use it in testing the logical prediction from §6.5, which states that the transition between two modes can reveal a discontinuity. This prediction may be best tested in a system reduced to its most elemental form, where the identification of modes can be relatively straightforward, in comparison with complex macroscopic systems as were explored in §6. Second, we use the conclusions to argue that the very appearance of the discontinuity—in this case the one entailed by the quantum *measurement problem*—is directly predictable from Theorem 1.

Theorem 1 ties together three principal concepts in physics that are common to all of its branches: frequency, time, and determinism. The relevance of the three to quantum physics is straightforward

[64] Ambiguity and double meaning is retained for a longer time than is implied by the above cited tasks, so a deterministic interpretation is most generally reinforced over time.

to demonstrate. Beginning from frequency, as was briefly discussed in §4.1.1 (and throughout §4), quantum mechanics has the concept of frequency at its heart. Both the Planck formula for the energy of a photon (Eq. 65) and the de Broglie formula for the momentum of a mass (Eq. 66) relate directly to the wave properties of all particles, namely, to their frequency—the temporal frequency to energy and spatial frequency to momentum. Thus, by extension, all quantum energy and momentum expressions can be ultimately related to frequency. Moreover, the reciprocal relations between conjugate quantities through the Fourier transform are central to the theory, and they embody the uncertainty principle, as well as the relationship between the position-space and momentum-space representations of the quantum state.

The role of time and determinism, or the lack thereof, is also central in quantum mechanics—as in all dynamical theories—but their exact interrelation has given rise to several conceptual conundrums that do not appear in classical mechanics. Of key interest to us is the measurement problem, which is one of the longest-standing open questions in quantum mechanics, and arguably its most foundational one (Wallace, 2008). It has to do with the incongruent description of the quantum system before and after a measurement has taken place—a *deterministic* rule characterizes the quantum system before a measurement, whereas a probabilistic rule accounts for the measurement outcome, as is roughly sketched below[65].

The quantum system is described by the *quantum state*, which contains the complete information about its dynamics. The values of all measurable quantities (*observables*) can be obtained by applying *Hermitian operators* on the state function[66]. Before any measurement takes place, the quantum system state, represented by the quantum *wave function* $\Psi(\boldsymbol{r},t)$, evolves in time according to the *Schrödinger equation* (Schrödinger, 1926), which contains the explicit total energy operator acting on the wave function

$$\left[\frac{\hat{p}^2}{2m}+\hat{V}(\boldsymbol{r},t)\right]\Psi(\boldsymbol{r},t)=i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) \tag{67}$$

where $\hat{V}(\boldsymbol{r},t)$ is the potential energy operator that is specific to the problem. The operator $\frac{\hat{p}^2}{2m}$ corresponds to the kinetic energy, which is dependent on the mass m and on the *momentum operator* $\hat{p}$, defined as $\hat{p}=-i\hbar\nabla$ in three dimensions, or $\hat{p_x}=-i\hbar\frac{\partial}{\partial x}$ in one dimension. Thus, the left-hand side of the equation is the total energy in the system, or its *Hamiltonian*. In three dimensions the *Hamiltonian operator* is therefore

$$\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+\hat{V}(\boldsymbol{r}) \tag{68}$$

Being a linear equation, Eq. 67 can be cast as an *eigenvalue problem*, where the Hamiltonian $\hat{H}$ is the energy operator acting on the state—an *eigenstate*—whose *eigenvalue* solution is the (real) energy level E of the state, or

$$\hat{H}\Psi=E\Psi \tag{69}$$

More generally, the solution to the Schrödinger equation is a superposition of eigenstates, each of which corresponds to a different eigenvalue[67]

$$\hat{H}\Psi_n=E_n\Psi_n \tag{70}$$

When the potential energy is time-independent $\hat{V}=\hat{V}(\boldsymbol{r})$, the energy is conserved and Eq. 67 may be solved using separation of variables, which results in solutions of the form

$$\Psi_n(\boldsymbol{r},t)=\psi_n(\boldsymbol{r})e^{-iE_nt/\hbar} \tag{71}$$

The general solution is a linear combination of states of the form of Eq. 71,

$$\Psi(\boldsymbol{r},t)=\sum_n c_n\psi_n(\boldsymbol{r})e^{-iE_nt/\hbar} \tag{72}$$

[65]Numerous texts provide rigorous introductions to the theory of non-relativistic quantum mechanics. The present account is primarily based on Griffiths and Schroeter (2018) and Cohen-Tannoudji et al. (2020).

[66]Physically, this is the necessary condition that ensures that the observable that is represented by the operator produces a real *expectation value* (i.e., an average value of an observable of a particular state). Algebraically, a Hermitian operator $\hat{A}$ is a linear transformation on a complex vector space, for which the inner product of vectors v and u is independent of the order of operation $\langle u|\hat{A}v\rangle=\langle\hat{A}u|v\rangle$.

[67]For simplicity, we only present the case in which the eigenvalues are unique (*non-degenerate*). The case with non-unique (*degenerate*) eigenvalues can have some differences. Also, for convenience we refer here only to discrete solutions, which are determined by potential problems with finite boundaries. Some systems such as a free particle (i.e., when $\hat{V}=0$) result in continuous solutions for which the resultant mathematics is somewhat different. However, the conceptual points in the discussion hold in all cases.

where c_n are coefficients that are determined by the specific potential energy of the system. It is evident that the Schrödinger equation itself describes the time evolution of the quantum system. For the time-independent energy-conserving Hamiltonian—by far the most useful case in the theory—*the time evolution operator* $\hat{U}(t)$ is defined through its action on the state $\Psi(\boldsymbol{r},t)$

$$\hat{U}(t)\Psi(\boldsymbol{r},t) = e^{-it\hat{H}/\hbar}\Psi(\boldsymbol{r},0) \tag{73}$$

which relates any moment $t \neq 0$ in the state's past or future to the state $\Psi(\boldsymbol{r},0)$ at $t = 0$.

The unusual aspect of these quantum states is that while they mathematically seem like standard 3D waves, they do not correspond to a physical wave, but to a probability amplitude at a particular time, which "lives" in an abstract *Hilbert space*[68]. In an orthonormal basis, which is always available for the solutions, this entails the normalization condition

$$\sum_n |c_n|^2 = 1 \tag{74}$$

The condition is unchanged over time through a property of the time evolution operator called *unitarity*. Each amplitude in the superposition state corresponds to one stationary eigenstate (with a particular n) of the form of Eq. 71 that appears, through measurement, with probability $|c_n|^2$. This is referred to as the *Born rule* (Born, 1926). Thus, the measurement can produce only one eigenvalue at a time in a probabilistic manner, corresponding to a particular eigenstate. This is a separate effect that is not captured by the time evolution of Eq. 73. However, given that the eigenvalue can only be determined by a measurement, whereas the superposition quantum state itself is not directly measurable, there is some ambiguity regarding what the mathematical state corresponds to (see §7.6). The measurement process itself depends on an unspecified measurement device, which is briefly coupled to (i.e., entangled with; Busch, 2009; see § 8.2.1) the quantum system at the instant of measurement, whereupon it produces a value that exists in the macroscopic (classical) domain and can be registered by an observer.

The moment when the two rules switch and a single eigenstate appears instead of the superposition state has been called the *collapse of the wave function* or *the reduction of the wave packet*. The Born rule cannot be obtained from Schrödinger equation itself and is conventionally taken as one of the postulates of quantum mechanics, proven only by experiment. Over many measurements, the eigenvalues of the ensemble of particles measured will be distributed according to their relative amplitudes (Eq. 74) that together constitute a *mixed state*. The wave function collapse is also associated with the apparent transition to classical physics with observations that remain unvarying, which take over from the quantum probabilistic domain.

Many proposals on how to solve the seeming paradox of the measurement problem have been attempted, either in the form of various *interpretations* to quantum mechanics as a whole (e.g., Jammer, 1974; Bub, 1999; Genovese, 2010; Tammaro, 2014; Drummond, 2019), or as more concentrated efforts to derive the Born rule from other postulates (e.g., Vaidman, 2020). The measurement problem is inherent to the standard interpretation that has been taught and practiced for over a century—*the Copenhagen interpretation*—which was primarily advocated by Bohr and Heisenberg (Stapp, 1972; Howard, 2004). Other interpretations have variably attempted to explain away the apparent dichotomy between the *deterministic* and statistical descriptions, and to account for the wave function collapse. In some cases, an additional goal of the interpretation has been to demystify or eliminate the role of the observer that may seem necessary in initiating the measurement in some versions of the standard Copenhagen interpretations (see §7.6).

It is not the intention here to review the measurement problem or its interpretations in any depth that will do justice to the decades-long work that has been put into it (Genovese, 2010). Rather, we would like to argue that the very existence of this problem is a corollary of Theorem 1 and as such it is inherent to quantum mechanics and is inevitable in its standard formulation that makes use of time and frequency. The same basic conclusion was arrived at by Bassi and Ghirardi (2000), who proved that the measurement problem inevitably arises for quantum systems that linearly evolve according to Schrödinger equation, and which are measured by a macroscopic apparatus with almost orthogonal states that correspond to its quantum state reading. While the present proof approaches the problem from an entirely different angle, it shares some things in common with Bassi and Ghirardi (2000): the pre-measurement state relies on the Schrödinger equation, and the post-measurement state alludes to the deterministic reading of the apparatus.

[68]Hilbert space is defined as a complete inner-product space, which in the quantum mechanical context refers more narrowly to the set of square-integrable functions in L^2 (see Footnote 19).

In order to prove these ideas, we will demonstrate how quantum theory of the pre-measurement state is formulated in P1, whereas the measurement output is produced in P2. Then we will invoke Theorem 1 to argue that the two cannot be simultaneous. We will conclude by briefly mentioning how some of the interpretations attacked the problem from different angles that pertain to different aspects covered by the theorem (§7.6).

7.2 The pre-measurement state

The approach followed in this subsection and the next is to find out how the pre-measurement and post-measurement states are mapped to the different modes of Theorem 1. A parallel argumentation may be based on the idea that the quantum state is isolated pre-measurement and it opens up to the environment post-measurement. As it is difficult to make statements about quantum theory that are not controversial whatsoever, some of the subsequent arguments may be inadvertently taken as belonging to a novel interpretation in its own right. Inasmuch as it may appear so, this argumentation adheres to the standard Copenhagen interpretation.

Time In the Schrödinger equation, where the energy is conserved (i.e., when the potential is time-independent), time is parametric[69]. Its wave-function solutions (and their superposition) are all stationary, so their time dependence is non-dimensional by definition (§3.5.3 and §3.5.5). The parametric nature of time in quantum mechanics is a corollary of *Pauli's "theorem"*, which goes back to a footnote that stated that there is no time operator in quantum mechanics, where time is only a number—or, as has been later elucidated, at least not a universal time operator that can be used in arbitrary time-measurement contexts[70] (Muga et al., 2002; Pauli, 1958 / 1980, p. 63). In more general presentations, it has been specifically emphasized that time is not a dynamic variable (Peres, 2002). In the framework of the present paper, this means that time appears in the pre-measurement equations in a parametric way that cycles the dynamical system states by virtue of the time evolution operation (Eq. 73), but it does not function as a dimension[71].

Frequency As for frequency, it always appears as a parameter in quantum mechanics and never as a variable, as long as the potential is time independent. There has been some outspoken discomfort with respect to the notion of *quantum jumps* between energy levels in discrete quantum systems (Born, 1926; Schrödinger, 1952a,b; Bell, 1987), which appear instantaneous—unlike any other physical systems that must take finite time to move between levels. These jumps correspond to a single-frequency photon emitted or absorbed in as a result. In contrast, a non-instantaneous jump could imply a continuous time-dependent function of frequency (or momentum). A recent measurement showed that the trajectories in Hilbert space of the quantum jumps are in fact not instantaneous if measured with very fine time resolution, although no time–frequency measurements accompanied these data to ascertain whether the frequency is indeed constant (Minev et al., 2019). See §A.2 for an in-depth discussion about this point.

Indeterminism Finally, there is the issue of indeterminism, which hovers over all of quantum theory and must be introduced into its formalism at some stage. Indeterminism at the pre-measurement stage can be argued for in different ways, but because of the *deterministic* time-evolution law, one way or the other our conclusion may be more controversial. While the Schrödinger equation itself (and specifically the time-evolution operator that is derived from it) is *deterministic*, its solution—the wave function itself—generally applies to ensembles of particles and was given the canonical interpretation of a *probability wave* by Born (1926). Perhaps no-one better than Born himself was to phrase the confusing nature of Schrödinger equation that is mathematically *deterministic*, and yet relates to an *indeterministic* measurement: "*One can perhaps summarize this, somewhat paradoxically, as: The motion of the particle follows the laws of probability, but the probability itself propagates in accord with causal laws.*" Therefore, observables, calculated as expectation values (see Footnote 66), relate to ensembles of particles, whereas the validity of the state for single particles

[69] It is an autonomous equation; see Footnote 11.

[70] This is the so-called *time problem* of quantum mechanics. For specific challenges to Pauli's theorem, see Galapon (2002) and Maccone and Sacha (2020).

[71] A similar assertion about the nondimensionality of time during the pre-measurement state was gathered by its "*timelessness*"—a property which is hallmarked by the reversible nature of the state that is compartmentalized between the state preparation and the collapse of the wave function (Thomsen, 2021). That analysis suggests that time exists outside of the pre-measurement state, as is also suggested below.

(their existence in their wave form) is questionable (e.g., Griffiths and Schroeter, 2018, p. 16; however, different interpretations disagree here; see §7.5 and §7.6). Thus, despite the *deterministic* law of the ensemble probability, the pre-measured state is *indeterministic*—we can obtain no certain information in practice about the pre-measurement particle position and momentum—only about ensembles in superposition.

These pre-measurement state characteristics cover the bulk of standard quantum physics, which maps it clearly to P1. In fact, P1 is overdetermined here, because according to Theorem 1, it is sufficient to know that time is parametric, or that frequency is parametric and that the system is *indeterministic*, or that its indeterminism is of the probabilistic kind (known horizon, but unknown vicinity). Either one of these indicates that the pre-measurement state must be at P1.

7.3 The post-measurement state

In contrast to the pre-measurement state, in the post-measurement state the system is moved to the (macroscopic) classical domain, where the other systems lie: the detector, observer, and farther environment. The logic below runs parallel to that in the P2 part of the theorem proof in §5.

Time The moment in which the measurement is carried out is distinguished: it is unlike any other point in time and has an absolute status, which means that, at this moment, time has become dimensional: the closed-quantum-system stationarity has been disrupted and it now becomes causally attached to the measurement system and environment. Thus, post-measurement time is necessarily dimensional.

Frequency The observed values of the frequency or spectrum (to the extent that they are observed) are parametric, as before, as they are fixed and not variable at the moment of measurement. They correspond to the particular (mixed) quantum state energy levels that made it to the classical domain through the measurement, after the superposition wave-function collapse.

Determinism The measurement result (after the collapse of the wave function) is definitive and stable—its value is no longer *indeterministic*, but has been *deterministically* measured with probability $p = 1$, to within a certain error margin defined by the fidelity of the measurement setup, experimental method, noise level in the experiment, etc.

These considerations clearly set the measurement reading—really the post-measurement state—within P2. Once again, though, the identification of P2 is overdetermined since according to Theorem 1, it is sufficient to know that the post-measurement state is deterministic, or that it is characterized by dimensional time and parametric frequency.

7.4 The instant of measurement and Theorem 1

According to Theorem 1, P1 and P2 are logically incompatible and cannot be simultaneously applied to Reality. This is true even if we consider the instant of measurement in which the quantum system and measurement apparatus have to be coupled—an entangled state that we will later consider to belong to P3 (see §8 and specifically §8.3.3)—which involves additional discontinuity between modes. Therefore, when switching between the two modes, as is the case between pre- and post-measurement quantum states, we can expect a discontinuity of an unspecified physical nature. Hence, insofar as the quantum states correspond to P1 and P2, the measurement problem appears to be inherent to quantum theory. This suggests that it may not have a solution, as long as time, frequency, and determinism are all used to formulate this physics, whether explicitly or implicitly. A solution to the problem, should one exist, may require a more fundamental change in the basic concepts as appear in the theorem and are carried over to quantum theory.

7.5 Qualifications

At least two of the assumptions made in the pre-measurement mapping may be challenged. First, the Schrödinger equation need not have a time-independent potential energy. A time-dependent potential energy can significantly complicate the solution, as it may render the equation dynamically non-autonomous and non-conservative, and the time dimensional, which generally leads to solutions that are nonstationary. Relatively simple potentials that include small-amplitude periodic (AM) or random (broadband) terms in the external potential are reviewed, for example, in Cohen-Tannoudji et al. (2020, pp. 1303–1355). In the AM case, the transition probability itself (say, between the initial and final states of a system with non-degenerate and discrete spectrum) depends on the external modulation frequency, whereas in the broadband case there may be additional energy level shifts in the different states due to fluctuating potential (e.g., as caused by bandlimited external electromagnetic fields). In any case, these problems are still solved using parametric time and frequency (now playing a more dynamic role)—but within the context of a probabilistic system—and are thus still generally mapped to P1 (see also comment on p. 7 of Thomsen, 2021). However, with some potentials, given that spectral considerations now play a key role in the analysis, it may appear to be more appropriate to map the pre-measurement states to P3 instead. One such implicit example may be hinted in Cohen-Tannoudji et al. (2020, p. 1329–1330), where the system described exhibits only "*little memory*" of its proximate time window dynamics, but not necessarily to more remote states in the past or future[72]. See also §8.

A second assumption we made above is that the Schrödinger equation always relates to ensembles of particles, where the statistical interpretation of a probabilistic pre-measurement state is warranted. However, quantum theory has been put to a test also on a single-particle scale, in which case Born's probability-wave notion becomes suspect, while the wave function description tends to apply, albeit counterintuitively, without resorting to the entire ensemble of particles and measurements to obtain correspondence with the wave function (e.g., Ringbauer et al., 2015). Most relevantly here, several studies specifically probed the pre-measurement state using single-particle measurements, by employing methods that do not lead to wave-function collapse (at least not immediately). Measurements were reported, for example, of single-particle superposition of eigenstates (Piacentini et al., 2017), simultaneous detection of a particle in two places at once (Zhou et al., 2017), "*strange*" (*weak*) eigenvalues that are not permitted post-measurement (Goggin et al., 2011), and complex-valued states (Lundeen et al., 2011).

These single-particle results beg the question of whether the single-particle pre-measurement state is indeed probabilistic as the results from Goggin et al. (2011) suggest, or it is in fact deterministic, as the (*deterministic*) Schrödinger equation prescribes and predicts and is suggested from Piacentini et al. (2017) and Zhou et al. (2017). If a deterministic pre-measurement state is indeed accessible, it may appear to be in violation of Theorem 1 and in contradiction of our falsification of P7 (§5), which states that Reality cannot have parametric time and frequency and also be deterministic. However, a close inspection of the abovementioned studies reveals that it is not at all clear that any of the stages of the measurement maps to P7 (see also argument for the impossibility of P7 in §A.2). First, a weak-measurement is still a measurement and entails a degree of coupling, however small, to the measurement apparatus, which results in a correspondingly weak perturbation of the pre-measurement state (Hance et al., 2023). Furthermore, in all cases mentioned above, the method involves a *post-selection* stage, which partially de-randomizes the otherwise probabilistic identity of the post-measurement state and thereby enables inference about its pre-measurement state, given sufficient averaging that reduces the statistical uncertainty (Aharonov et al., 1988). Therefore, while the system response is made to appear deterministic, its output is nevertheless drawn from a probability distribution, and as such it requires an ensemble of similarly post-selected single particles, with corresponding averaging. A deterministic interpretation of the post-weak-measurement, on the other hand, would have each individual measurement distinguished in time, but mired in noise, which requires sufficient averaging to obtain a clear signal about the underlying value of the pre-measurement state—still P2.

The aforementioned single-particle experiments provide some insight on the pre-measurement physics and the validity of quantum theory, but due to the complex combination of methods and underlying analytical assumptions, they are not amenable to a straightforward interpretation (e.g., Vaidman, 2017; Hance et al., 2023), including the classification according to the framework set in this

[72] More complex time-dependent potentials have also been explored in the nonlinear dynamical and mathematical physics literature and are beyond the scope of this discussion. However, we can only note that similar to the complex nonstationary examples referenced in §6.3, it is not always trivial to place the analysis within a single mode, because different mathematical methods may be used in succession to reach a solution. This means that in some cases it may appear that a *deterministic* approach of P2 is used to solve an essentially indeterministic equation, in which case care must be taken to elucidate the exact status of both time and frequency as parameters or dimensions. Also, assumptions regarding complete knowledge of the future or past potential, as well as neglecting of various transient states, must be considered.

work. It is not impossible that future methods for single-particle measurements will be simpler to interpret in a way that could challenge the above interpretation and be more at odds with Theorem 1. At present, however, the validity of the theorem seems to hold.

7.6 Alternative interpretations

Different interpretations of quantum mechanics have aimed to address the seeming paradoxes and inconsistencies in the orthodox Copenhagen interpretation, key to which is the measurement problem. They tend to rely on the existent formalism of quantum theory and do not necessarily add any new elements to the theory, but rather offer a fresh perspective that, at its finest, may enable the circumvention or dissolution of the measurement problem. As of the time of writing, no interpretation is universally accepted by the physics community (Schlosshauer et al., 2013; Sivasundaram and Nielsen, 2016). Furthermore, all major interpretations appear to substitute some issues with other issues, which only makes them differently deficient than the standard interpretation (Tammaro, 2014). For some, this very limited success should amount to an altogether abandonment of the notion of (re-)interpretation of the theory, especially given its undeniable success in its standard form, despite these outstanding issues (Fuchs and Peres, 2000). For others, the current impossibility to reject incorrect interpretations in experiment is both unreasonable and frustrating (Cabello, 2017). As this work's main focus is not quantum theory, the various interpretations are mentioned here without review and with minimal commentary. Moreover, they mostly appear with reference to their original versions and not to subsequent iterations. Rather, a brief overview is provided of how a subset of the major interpretations can be understood through the lens of Theorem 1.

Several models tackled specifically the instant of measurement that can provide an account of the discontinuity. What may be one of the most controversial interpretations is that the wave function collapse is caused by a conscious agent that performs the measurement. The idea was implied by von Neumann (1932 / 2018) in his seminal analysis of the measurement process, alluded to by London and Bauer (1939 / 1983), and was finally made explicit by Wigner (1961 / 1983). This interpretation conforms to the standard Copenhagen interpretation, with the uncomfortable addition of the nonphysical, conscious observer that is responsible for the switch between the pre- and post-measurement states (P1 to P2).

Other interpreters reacted by explicitly trying to eliminate the role of the observer. The *objective-collapse model* or *GRW model*, named after the original authors (Ghirardi, Rimini, and Weber, 1986), posits that particles *spontaneously collapse* according to a hypothetical random process, at an infinitesimally slow rate that is suggested to be a constant of nature. This would statistically render isolated quantum systems effectively immune to collapse. However, once they get entangled with the measurement apparatus, the large number of particles that make the combined macroscopic system results in a propagated collapse, because the rate is now scaled by the number of particles. This theory stitches the pre- and post-measurement processes, by deferring the primary indeterministic component to the spontaneous collapse itself. This would make it a P1 to another P1 to P2 model. While the role of the observer is eliminated, the discontinuity is not. In *the transactional interpretation*, the measurement is decomposed into *retarded* and *advanced waves* in time that are passed between an *emitter* and an *absorber* (for example, where one can assume the measured particle and the other is the apparatus) (Cramer, 1986). The emitter and absorber perform an instantaneous "*handshake*"-like sequence of wave exchanges that analytically lead to the Born rule. Despite its unintuitive mechanism the process is causal and may be framed as a nonstationary P3 process. In any case, the effect of the (unmeasurable) intermediate handshake process reproduces the P1 to P2 pre- to post-measurement states.

Some interpreters focused on making the theory fully deterministic, thereby eliminating all the otherwise inexplicable randomness. *Bohmian mechanics* or *pilot wave theory* (Bohm, 1952a,b; anticipated by de Broglie, 1930 and Rosen, 1945) exploits a decomposition of the wave function into two parts (amplitude and phase), which can be roughly associated with a *deterministic* particle and a probabilistic field. While we do not have the initial conditions for the particle (considered its "*hidden variables*"), having them would result in a fully *deterministic* (classical-like) solution pre- and post-measurement, that is only briefly perturbed during the measurement, which is nonstationary in nature. Given the hypothetical access to the time-distinguished initial conditions, this model can be classified as a P2 to P2 process, perhaps going through P3 due to the perturbation, but which can otherwise remain continuous. Hence, the measurement problem may be eliminated here, at the initial cost of having to deal with unmeasurable hidden variables. A completely different interpretation that is geared to achieve the same continuity is *the many-world interpretation* (Everett III, 1957). Here, instead of collapsing, the wave function *branches* to multiple "*worlds*" upon every measurement—each world corresponding to one eigenstate contained in the superposition pre-measurement state—although we (somehow) experience only a single branch of all the branches that continue to exist post-measurement. The branches depend on an unspecified memory that has some bearing to past states. Considering this solution from time zero before any state began branching, this interpretation achieves full determinism, in line with P2, at the cost of a significant loss of apparent realism. The related *many-minds interpretation* by Zeh (1970) is perhaps an amalgamation of the consciousness-collapse and the many-worlds interpretation, assigning the conscious observer with the task of isolating or selecting a single state out of the superposition of states that arrive at the measurement apparatus after measurement.

Other interpretations embraced the indeterministic element in quantum theory. Ballentine (1970) proposed the *ensemble interpretation*, according to which the quantum state is nothing but a description of statistical ensembles of many particles and, as such, it is subjected to statistical rules, both pre- and post-measurement[73]. The collapse is only apparent and the Born rule (as well as a host of other statistical properties) is a reflection of the probabilistic nature of the state. In all cases, time is a parameter of the ensembles. Thus, this interpretation remains in P1 pre- and post-measurement, and the measurement problem is avoided. According to the *QBism* interpretation, indeterminism is interpreted using the definition of *Bayesian probability*, which assigns values of belief to different possibilities of the system and are therefore particular to the observer (Fuchs, 2010; Fuchs et al., 2014). The Born rule is therefore a direct application of this logic, as each outcome occurs at a different likelihood. However, once the observer obtains the result, it essentially becomes deterministic for them and can be incorporated into their experience. Therefore, QBism still retains a P1 to P2 transition and while the discontinuity is justified, it persists. A different probabilistic take on the measurement process is the *consistent histories* interpretation, which models the probability associated with different sequences of events (time-specific states) that may take place in the experiment through to the pre- and post-measurement moments (Griffiths, 1984). Each *history* to its own may be perhaps taken to be deterministic, while the totality of all of them produces a P1 Reality both pre- and post-measurement, and is therefore indeterministic. However, the measurement is a conditional event within the particular history and no collapse takes place. Finally, in the *relational interpretation*, the relative information available to the system and the observer is used as the guiding principle (Rovelli, 1996). Here, the measurement is a querying action by the observer, which is any system whatsoever defined by its relative boundaries to the observed system. The interpretation is fully within P1, given that everything (including the states themselves) is relative, although the measurement constitutes a disruption that may be taken to be P3, as in some of the above interpretations.

Finally, we should also mention *decoherence*, which is the logical continuation of the measurement instant that appears as an instantaneous collapse of the wave function, as the system and apparatus couple with the whole environment in which the measurement takes place (Zurek, 2002). Decoherence describes the diffusion of the superposition state reduction and effective appearance of a mixed state over a very brief, albeit finite, time course. Because it is essentially a continuous process, it is possible to add decoherence to different interpretations in their own modes. So it offers a path of interpolation the P1 to P2 transition in the standard interpretation, while remaining in P2 in the case of the many-worlds interpretation.

In conclusion, most interpretations above recognize the shift between a closed pre-measurement system that opens up and gets entangled with the apparatus post-measurement, whereupon it becomes merged with the observer, the environment, or even with the whole universe. In general, this shift in boundaries is neither well-motivated nor well-understood. Ideally, it is explained as something that "just happens" by virtue of coupling between the closed microscopic and larger macroscopic systems that somehow results in the selection of one particular state outcome. Less ideally, it requires the involvement of an unquantifiable, nonphysical element of consciousness. The indeterminism may be a necessary evil, a reflection of the observer's ignorance or, perhaps, the insufficient reach of the theoretician. Short of being inherent to the physics, some believe that apparent indeterminism should be replaceable with a fully deterministic account, à la Laplace (see § A.1), which begs the question—what amount of realism is a plausible tradeoff in exchange for banishing indeterminism[74]? Interpretations that might seem to be doing away with the measurement problem

[73] Ballentine (1972) argued that this interpretation corresponds also with Einstein's view of quantum mechanics.
[74] Granted, some may argue that realism is already a rare commodity in the quantum domain.

are those that manage to retain the system within either P1 (indeterminism) or P2 (determinism) over the pre- and post-measurement system[75]. At any rate, none of the interpretations reviewed seems to violate Theorem 1, as the discontinuity between P1 and P2 may be either accounted for or eliminated by maintaining a continuous P1–P1 or P2–P2 mode. Note that none of the mentioned interpretations has directly appealed to considerations of time or frequency as parameters or dimensions.

8 Example VI: Nonlocality

8.1 Introduction

Up until now, all the examples discussed could be considered to be of strictly epistemic nature, which represents either a mathematical convenience in analysis and method, or some inherent definitional oddities related to the interrelationship between frequency, time, and determinism. Even in the motivating case of perception, where the role of frequency is demonstrably key (§ 2.1), the lack of uniformity between the qualia associated with the different senses and their respective frequency ranges may be thought of as a representational tool of reality for or by consciousness, rather than corresponding to an objective dimension of Reality. Hypothetically, inconsistencies that arise in invoking these concepts may be accounted for by using more adequate definitions than are currently available, as well as demanding more careful differentiation between the physical Reality and perceptual reality—something which this work has been partial about from the get go (§1).

The idea presented in this section goes further than the previous examples by serving as a gateway for the possibility of an ontic frequency dimension—a dimension that is an actual part of physical Reality, and yet it exists outside of space and time. This ontological view can nonetheless coincide with the concepts that are encapsulated within Theorem 1. While undoubtedly deemed as both speculative and unrealistic (or even downright non-physical), given the already well-established science of quantum nonlocality, such an ontological reach of the theory may not be all that far-fetched.

The distinction between local and nonlocal effects in physics has acquired prominence of late, for a large part due to the inherent nature of quantum mechanics. *Locality* entails that an effect measured on a system must be caused by events or forces (perhaps represented by a field) in the immediate vicinity of the system itself, in its own position and time coordinates. In contrast, *nonlocality* has an effect depending on something that takes place elsewhere—at a region of space or at a moment in time different and disconnected with where the effect action itself is located. Another requirement for locality is obtained by imposing special relativistic constraints, which limit the velocity of any communication to the speed of light (according to the special theory of relativity). This leads to a stringent notion of causality: nothing can cause itself—no action can travel faster than the speed of light so to change its own past (i.e., retrocausality is prohibited). Hence, in applying causality as a constraint, anything that appears to violate it may be suspect as nonlocal.

Nonlocality in quantum mechanics has been primarily associated with *quantum entanglement* and the *Einstein–Podolsky–Rosen (EPR) paradox* (Einstein et al., 1935) and with its subsequent benchmark in the form of *Bell theorem* and *Bell tests* (Bell, 1964; see Fig. 20). Although the existence of nonlocal effects through entanglement has been repeatedly demonstrated in experiments since Bell, a consensus interpretation of these measurements—much like regarding other aspects of quantum theory—does not exist as of yet. Whenever *loopholes* had been discovered in the experimental protocols of particular Bell tests or their associated theory, the local-variable explanation is generally preferred and may be conjectured to override any nonlocal account. However, Bell theorem ruled out any *local* hidden variable explanation in the observed case of violation of the Bell's inequality, which has left room only for a *nonlocal* hidden variable theory (Bell, 1975). Still, it is not at all clear that even that may be within reach (Leggett, 2003; Ringbauer et al., 2016; Dalton, 2024).

The objective in this section is to briefly review the main nonlocal effects in quantum mechanics—entanglement in Bell tests being the primary one—alongside mentions of the Aharonov–Bohm effect and the two-slit experiment as additional examples. Then we explore whether the nonlocality inherent in the frequency dimension and in quantum mechanics may be amalgamated by applying Theorem 1 to the problem. Answering in the affirmative, we shall further inquire whether the frequency dimension could provide a full solution to the nonlocality problem. It is therefore suggested

[75] While this can be seen as an obvious advantage of some interpretations (e.g., Bell praised Bohm's theory because it eliminated "*...any need for a vague division of the world into 'system' on the one hand, and 'apparatus' or 'observer' on the other*;" Bell, 1984, p. 1), others consider it a failure to reproduce the measurement problem or the measurement outcome (Tammaro, 2014).

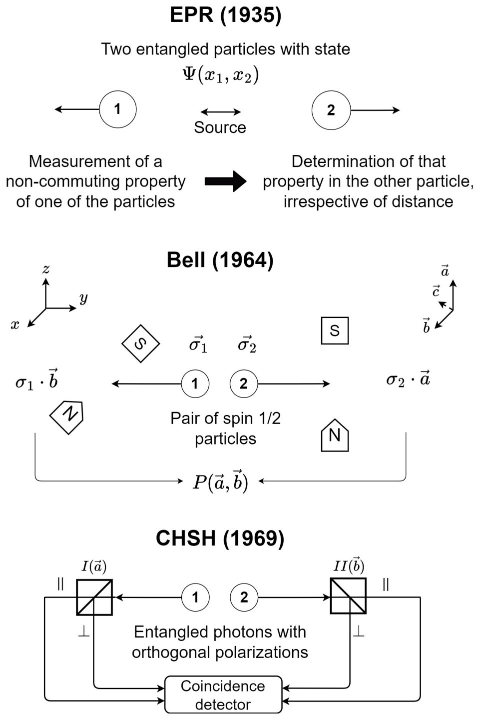

Figure 20: Three theoretical realizations of the quantum nonlocality paradox. **Top**: The original Einstein–Podolsky–Rosen (EPR) thought experiment looked at a generic system that is composed of two subsystems, which interact for a short duration, before they are separated (Einstein et al., 1935). When they are sufficiently distant, their interaction is null and a quantity that is tied to a noncommuting operator (i.e., momentum) is measured on one particle. According to quantum mechanics, this would also determine the same quantity in the other particle, appearing as a nonlocal action at a distance. **Middle**: Bell (1964) extended a thought experiment by Bohm and Aharonov (1957), in which two entangled spin-half particles (e.g., an electron and a positron after a decay of a neutral pion particle) are measured using two *Stern–Gerlach* apparatuses to determine their spin value (either up or down). Bell showed how if the two apparatuses are set to different arbitrary orientations ($\vec{a}$, $\vec{b}$, and $\vec{c}$), the three possible correlations between the measurements in the different orientations (designated with $P(\vec{a}, \vec{b})$, $P(\vec{b}, \vec{c})$, and $P(\vec{c}, \vec{a})$) can reveal whether the results are in fact determined by a hidden parameter, or whether the predictions of quantum theory are correct. **Bottom**: The *Clauser–Holt–Shimony–Horne* (*CHSH*) practical variation of the Bell test (Clauser, Horne, Shimony, and Holt, 1969) was the first to be realized and to conclusively violate Bell (CHSH) inequality in an experiment by Aspect, Grangier, and Roger (1982). CHSH replaced the spin-half pair with an entangled photon pair with orthogonal polarizations, as a result of an *atomic cascade* transition. Each photon goes through a *polarizer filter* (square with diagonal in the figures) that is set up at some angle, whose two orthogonal outputs (marked with $\perp$ and $||$) are detected and correlated with the outputs from the other polarizer, which is set to another angle. Using combinations of four different angles (two on each side), it is possible to benchmark a model that contains a hidden parameter, or obtain a different prediction that is in line with quantum theory and is nonlocal.

that both quantum mechanics and an ontic frequency dimension are nonlocal. We shall explore below whether the two nonlocalities coincide. While this discussion runs the risk of embracing the nonlocality notion a bit too enthusiastically for some critics, we can respond using an inversion of the beaten aphorism: extraordinary evidence requires extraordinary claims.

8.2 Hallmark quantum nonlocal phenomena

Quantum nonlocality has been brought to attention primarily through the EPR paradox and the subsequent work of Bell. However, its shadow has been persistently present at the very heart of quantum mechanics, beginning from the quantum measurement and wave function collapse, as was already implied indirectly by Born (1926): "*...one must show whether the interference of damped 'probability waves' suffices to explain the phenomena that apparently point to a coupling that does not relate to spacetime.*[76]"

[76]The original German text is: "...hier muß sich zeigen, ob die Interferenz gedämpfter, 'Wahrscheinlichkeitswellen' hinreicht, diejenigen Erscheinungen zu erklären, die anscheinend auf eine raumzeitlose Kopplung hindeuten." More nuanced references to nonlocality may be traced in earlier writings of Einstein (Doyle, 2015).

Nonlocality also features in a host of other more or less obscure quantum effects, which are typically distinguished from Bell's nonlocality, sometimes by differentiating between kinematic (related to the quantum mechanical Hilbert-space structure) and dynamic (related to the quantum equations of motion) types of nonlocality. As is briefly reviewed below, there is no consensus that nonlocality is the most appropriate explanation for some of these otherwise unintuitive results, as it has struck many, beginning from Einstein himself, as fundamentally unrealistic[77].

8.2.1 Entanglement and Bell nonlocality

When two or more quantum systems interact, they become correlated. If the correlation is such that the individual systems cannot be treated as independent, they are considered *entangled*. More technically, entanglement entails the *non-separability* of the component quantum states within the state of the combined system. The entangled state is distinguished from a *product state*, in which the component systems (the *subsystems*) retain their identity when the combined state function is separable, that can be factored as a product of the subsystem states. Specifically, in two-particle entangled systems, determination of the state of one of the particles (through measurement) predicts the state of the other particle. For example, the state of a system composed of two spin-1/2 particles (a *singlet state*), which has a total spin of zero, can be entangled using the spin as the degree of freedom (Bohm, 1951)—represented symbolically using the *bra–ket notation*, where $|A\rangle$ represents the state A. In spin entanglement, the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right) \tag{75}$$

where the index following each state represents the particle number 1 or 2 and the $\uparrow$ and $\downarrow$ represent its particular spin state, which can be either spin up or down. A measurement of one of the particles would result in a state of either spin up or spin down with probability 1/2 (according to the Born rule), in which case the other particle would necessarily result in spin down or spin up, respectively.

The remarkable thing about such entangled systems is that their correlation can hold remotely, even if the particles are separated in space over arbitrarily large distances. However, the final state of the particles is undefined (random) until the instant of measurement, which means that the particles have to somehow communicate the measurement between one another, what appears as instantaneous and perfect correlation between the particles. Such communication, however, if carried out through space, would be *superluminal* (faster than the speed of light), in violation of the special theory of relativity (Sommerfeld, 1914; Brillouin, 1914; Ghirardi et al., 1980, 1988). As there is no known field to constitute the correlational link in spacetime between the entangled subsystems, this property is nonlocal.

Despite the repeated demonstration of nonlocal correlation at ever-improving-controlled ("*loophole-free*") experiments, current theory does not provide an explanation for what it is in the physical Reality that is captured by the quantum state, which ends up resulting in nonlocality, aside from a few conjectures about the geometry of spacetime (see §8.2.4). Rather, the standard theory simply brings out the physical outcomes that are encapsulated in the intricate mathematical formalism underlying the quantum state.

Empirical tests of the nonlocality of entangled states were operationalized by Bell through his test and associated theorem (inequality) (Bell, 1964) and their subsequent variations (notably, Clauser, Horne, Shimony, and Holt, 1969; see Fig. 20). The original EPR argument was that quantum theory cannot be considered realistic, because it gives rise to "*spooky action at a distance*," as Einstein wrote to Born later (Born et al., 1971, p. 158)—despite the prediction made by quantum theory (Einstein et al., 1935). However, according to EPR, if such a correlation is measured between remote particles, then there must be a hidden variable that accounts for the correlation, which means that quantum theory itself is incomplete. The alternative proposed by EPR is that there is no simultaneous Reality that applies to any two noncommuting quantities (such as position and momentum), where Reality was defined, for the narrow scope of the EPR paper, by "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*" Bell (1964) reified this logic by producing two predictions of the results of a gedankenexperiment proposed by Bohm and Aharonov (1957). One prediction allowed for a "hidden variable" to account for the

[77]The roots of the discontents regarding *action at a distance* go back to Newton with respect to gravity. See Jansson (2024) for discussion and references.

hypothetical nonlocal correlation and was expressed as an inequality, whereas the other prediction was obtained using standard quantum theory that contains no hidden variables (i.e., implying that quantum theory is indeed complete). If the latter prediction is correct, then it violates the inequality of the former—something that could be (eventually) tested experimentally. Hence, experiments that violate the Bell inequality are considered to follow quantum theory that is complete, where no local hidden variables play a role. The theorem, though, left room for nonlocal hidden variables to predict the results (Bell, 1975).

Despite the consistent, high-quality outcomes of the various Bell tests in supporting quantum theory (i.e., the existence of nonlocal correlations with no hidden local variables), beginning from Aspect, Grangier, and Roger (1982) and culminating in Hensen et al. (2015), Giustina et al. (2015) and Shalm et al. (2015), there remain dissenting interpretations of the experimental results. They generally argue that the measured correlations are not indicative of nonlocality, but rather of other causes. The main motivation behind many of the alternative views has been to reinstate a realistic (i.e., local and causal) physics, alongside the elimination of what may be considered superfluous metaphysical baggage that tends to mire discussions about nonlocality (e.g., Griffiths, 2003; Bohm and Hiley, 1993, pp. 129–130). Roughly speaking, these are the main classes of interpretations of the Bell test results and apparent associated nonlocality:

1. Nonlocal correlations emerge from outside of spacetime (e.g., Gisin, 2009). These may or may not be the result of hidden nonlocal variables (Garuccio and Selleri, 1976; Leggett, 2003; Gröblacher et al., 2007; Dalton, 2024). Various loopholes in early Bell tests may have given the impression of nonlocality, but the physics could have been local. They are roughly classified into *detection loopholes*, which primarily entail low-efficiency detection and questionable methods to deal with no-event detections; *locality loopholes*, when the detectors are not sufficiently well-separated to preclude light-speed communication, or their randomly-triggered measurements are not truly independent of each other; and *finite statistics loopholes* that make unwarranted assumptions about the underlying probabilistic distributions of the measurements (Brunner et al., 2014). These loopholes seem to have been fully redressed over the last two decades, so unless further loopholes are going to be pointed to (and experimentally tested) the nonlocal test results are considered valid and nonlocality is real.

2. The nonlocality attributed to the Bell test results is erroneous for different reasons. For example, a lack of statistical independence of the two remote measurement setups and the role of the experimenters therein can render the test local (*superdeterminism*; Hossenfelder and Palmer, 2020). Another criticism is that the Bell test results must be viewed with appropriate consideration of quantum contextuality[78], which when correctly applied invalidates the claims for nonlocality (Khrennikov, 2019; Kupczynski, 2020). Finally, according to Griffiths (2020), interpretations that have led to the conclusion that nonlocality exists have been based on several errors such as misapplication of the wave function collapse concept, misuse of classical hidden variable models instead of quantum ones, and wrong logical reasoning in operationalizing the EPR paradox. It was argued that when these are all redressed, a fully local account of the results of Bell tests follows.

3. Superluminal correlation that does not yield communication may be possible according to the special theory of relativity, which may be sufficient to produce the correlation between entangled particles (John Bell, as part of an interview with Davies and Brown, 1986, pp. 45–57). Several models have examined possibilities for this solution that transforms apparent nonlocality to locality-of-a-sort (e.g., Eberhard, 1989; Bohm and Hiley, 1993; Scarani and Gisin, 2005). Experiments that tested these models have so far only resulted in establishing lower bounds for the superluminal velocity (Cocciaro et al., 2018), so instantaneous nonlocal correlations cannot be ruled out.

4. Attempts at challenging the notion of causality and its relationship to the special theory of relativity may open the door for local or nonlocal communication, which is conventionally seen as noncausal, or retrocausal (Hall and Branciard, 2020; Adlam, 2022).

Respectfully acknowledging the ongoing controversy, we shall strictly focus on the nonlocal interpretation (1), accepting the received version of quantum theory and loophole-free experimental outcomes.

[78] *Quantum contextuality* refers to dependence of the quantum measurement outcome of one observable on the other observables that are being simultaneously measured in the same experiment (Budroni et al., 2022).

8.2.2 The Aharonov–Bohm effect

In the Aharonov–Bohm effect, an electron beam is split into two and each beam is made to pass next to a solenoid, which is characterized by a magnetic flux F_0 within its volume at $r < 0$, but produces no magnetic field outside of it at $r > 0$ (see Fig. 21, right) (Aharonov and Bohm, 1959; the effect was anticipated by Ehrenberg and Siday, 1949). Although there is no magnetic field outside of the solenoid, there is a non-zero *magnetic vector potential* $\boldsymbol{A}$ there[79], which is related to the magnetic flux through

$$\oint \boldsymbol{A} \cdot d\boldsymbol{x} = \int \boldsymbol{B} \cdot d\boldsymbol{s} = F_0 \tag{76}$$

with $\boldsymbol{B}$ being the magnetic field and $\boldsymbol{s}$ is the surface through which the flux is calculated. The vector potential also appears (as an operator) in the Hamiltonian

$$\hat{H} = \frac{\left(\hat{\boldsymbol{p}} - \frac{q}{c}\hat{\boldsymbol{A}}\right)^2}{2m} \tag{77}$$

where $\hat{\boldsymbol{p}}$ is the momentum operator and q is the charge of the particle. After passing the solenoid in different paths, the electron beams are recombined and made to interfere. The resultant interference pattern is nevertheless found to be affected by the solenoid, as a corresponding phase shift becomes visible and depends on the vector potential

$$\Delta\phi = \frac{q}{c} \oint \boldsymbol{A} \cdot d\boldsymbol{x} = \frac{q}{c} F_0 \tag{78}$$

which corresponds to a phase dependence on the magnetic flux *within* the solenoid. The difficulty arises because the vector potential should have no direct physical role according to classical electrodynamic theory, which makes any dependence on it puzzling.

An analogous electric effect contains a similar setup where the electrons are made to pass through a time-dependent electric potential region with zero electric field (Fig. 21, left).

Both magnetic and electric effects are in violation of classical electrodynamics that attributes no effect to a local potential in the absence of a local field that is necessary for producing a force (Footnote 79). In quantum mechanics, a nonlocal interpretation is necessary if the effect is strictly dependent on the remote field, because the potential is, by definition, not gauge-invariant and cannot be observed. The alternative is that the potential has a local physical effect, which defies gauge invariance and contradicts classical electrodynamic theory and is, therefore, a no-less implausible a solution.

The first experiment to test the magnetic Aharonov–Bohm effect was presented by Chambers (1960), but it was not possible to rule out that there was no field leaking into the region where the electron beams traversed—a major challenge in testing this effect. To date, the main experimental evidence for the Aharonov–Bohm effect is widely considered to have been obtained by Tonomura et al. (1986). A confirmatory test of the electric Aharonov–Bohm effect is still pending (Weder, 2011).

Just as with the nonlocality associated with the Bell test results, so is a nonlocality that may underlie the Aharonov–Bohm effect seen by many as an implausible interpretation that has no place in physics that describes Reality. Therefore, various alternative explanations have been proposed that associate the Aharonov–Bohm phase shift with a local effect, of which only the bulk of the most recent suggestions are briefly mentioned. An analysis by Kang (2015) contested the nonlocality interpretation of the Aharonov–Bohm effect, as he was able to find a gauge-independent Hamiltonian to replace the standard one that is gauge dependent (cf. DeWitt, 1962; Aharonov and Bohm, 1962). In his alternative *semi-classical* analysis (i.e., combining quantum and classical elements in the model), he showed how at least in some of the experimental configurations, the electromagnetic field is not fully eliminated, rendering the phase shift local. Vaidman (2012, 2015) came up with an analogous problem statement and used semi-classical methods to obtain a local phase dependence.

[79] The electromagnetic field may be expressed as the derivative of a potential function, as part of its *gauge invariance* property. According to classical electrodynamics, there is freedom in selection of the *gauge* term, which does not impact the physics of the problem, but only represents a mathematical degree of freedom owing to vector calculus identities and Maxwell equations (e.g., Jackson, 1999, pp. 239–240). Thus, the magnetic field $\boldsymbol{B}$ satisfies $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, with $\boldsymbol{A}$ being the vector potential. The electric field satisfies $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t}$, where Φ is the scalar potential. These identities result in allowable transformations that do not impact the fields—both to the vector potential $\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$ and to the scalar potential $\Phi \to \Phi' = \Phi + \frac{\partial \Lambda}{\partial t}$, where Λ is some scalar function.

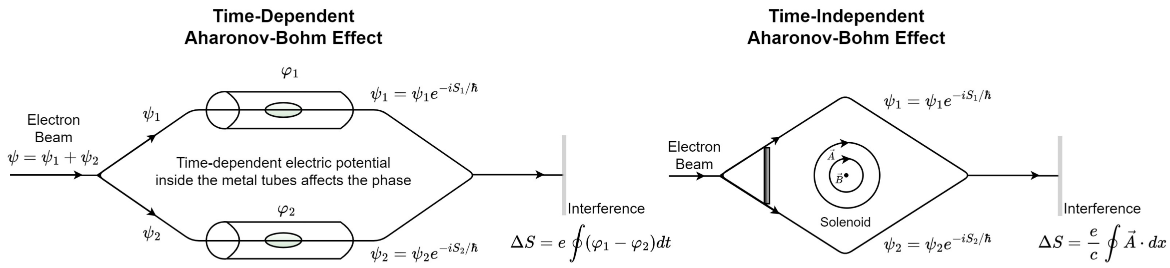

Figure 21: The two Aharonov–Bohm effects (Aharonov and Bohm, 1959). In both effects an electron beam is split into two different paths that are later recombined and form an interference pattern. **Left**: In the time-dependent effect the electrons are made to cross a zone with zero electrical field, but with a time-dependent scalar potential φ that is different between the two paths. **Right**: In the two electron paths there is a different time-independent vector potential $\vec{A}$ (denoted with $\boldsymbol{A}$ in the text), yet the electromagnetic field $\vec{B}$ (in the text, $\boldsymbol{B}$) is zero there, due to shielding by metal. In both effects, the asymmetrical potentials between the paths create a phase difference that is detectable in the interference pattern of the two beams after being recombined.

Pearle and Rizzi (2017) confirmed these results by quantizing the models of the electron, solenoid, and vector potential (as well as pair combinations thereof). Other explanations resorted to quantum electrodynamic approaches, whereby virtual photons mediate the interaction between the charged particle and the magnetic flux, according to the gauge field (Santos and Gonzalo, 1999; Li et al., 2018; Marletto and Vedral, 2020; Saldanha, 2021; Kang, 2022). For example, Kang (2022) obtained the magnetic Aharonov–Bohm phase using a local model with a general gauge (but not for the electric Aharonov–Bohm effect), where the scalar potential had a physical role and was gauge variant. Saldanha (2021) too proposed a general gauge-invariant explanation for general geometries using the concept of virtual photons, generated due to fluctuations in the vacuum electromagnetic field, which depends on the path of the charged particles[80]. While only a subset of these local models were addressed (or rebutted) in literature, a well-established alternative local model has not been obtained yet and the nonlocal interpretation remains dominant at present (Aharonov et al., 2015, 2016).

8.2.3 The double-slit experiment

Perhaps the most iconic of all (gedanken-)experiments of quantum mechanics, *the double-slit experiment*, is often invoked to demonstrate the *wave–particle duality* and quantum *complementarity* (Bohr, 1970; Scully et al., 1991). The starting point is the classical double-slit experiment with light that gives rise to an interference pattern on a screen, which is well accounted for using the classical theory of light waves (Young, 1804). When realized within a quantum setup—for a beam of particles (photons, electrons, neutrons, atoms)—it is shown that it is impossible to simultaneously determine in which of the two slits a particle has gone through without destroying the interference pattern. Hence, the instantaneous measurement can show either a particle-like or a wave-like behavior, but not both together. This effect has been attributed to the entanglement between the measurement apparatus and the particles (Dürr et al., 1998). It has been noted several times that the double-slit experiment is nonlocal in nature (Spasskiĭ and Moskovskiĭ, 1984; Chiao et al., 1995; Popescu, 2010; Susskind, 2016; Aharonov et al., 2017). Notably, both Popescu (2010) and Aharonov et al. (2017) underscored how the interference pattern depends on the relative phase between the states of the two possible paths

$$\Psi(\boldsymbol{x}) = \frac{1}{\sqrt{2}} \left[e^{i\varphi_1} \Psi_1(\boldsymbol{x}) + e^{i\varphi_2} \Psi_2(\boldsymbol{x}) \right] \tag{79}$$

with Ψ_1 and Ψ_2 corresponding to the particle state going through slits 1 and 2, which has an absolute phase designated by φ_1 and φ_2, respectively. However, the absolute phase of the quantum state is unmeasurable—only the relative phase shift $\varphi_1 - \varphi_2$, which can be shown to be nonlocal, because the particle passing through one of the slits depends on the potential in the other remote slit.

Here too, the nonlocal interpretation of the two-slit experiment has been challenged and alternative local models exist. For example, Catani et al. (2023) applies statistical restrictions on the available knowledge of the observer, which turn out to be consistent with a classical (and realistic)

[80]It should be noted that quantum field theory itself is not free from nonlocality, which is inherent in its measurement process formulation (Sorkin, 1993; Borsten et al., 2021; Papageorgiou and Fraser, 2024).

local interpretation of different variations of the interference experiment. In this interpretation, the physical Reality does not depend on the observer (i.e., whether it is the particle or the wave nature of the experiment that are being targeted in the measurement).

A different type of interference effect using an entangled photon pair is found with *intensity interference* cancellations (Brown and Twiss, 1956; Glauber, 1963), whereby a photon may be completely "forbidden" from certain positions, which would be allowed in an analogous classical setup (Ghosh and Mandel, 1987). Intensity interference (also called *fourth-order interference*) may be considered a nonlocal effect in its own right (Spasskiĭ and Moskovskiĭ, 1984) and is commonly used in entanglement experiments.

8.2.4 Accounts for entanglement nonlocality in quantum mechanics

While critics of nonlocality occasionally propose alternative local explanations that resolve the seeming paradoxical nature of the nonlocal results (§8.2.1, §8.2.2, and §8.2.3), there has been a relative paucity of nonlocal explanations that attempt to account for that mystery, which is otherwise implicitly built into the quantum mechanical formalism.

Perhaps the earliest explanation goes back to the Bohmian interpretation of quantum mechanics (Bohm, 1952a,b; see § 7.6), which fundamentally devises a nonlocal *quantum potential* that is a constituent of the wave function itself (Bohm and Hiley, 1975). The quantum potential appears to span the entire universe and is specifically able to connect entangled particles. However, the existence and validity of the quantum potential is in question. The nonlocality here is inherent to the higher-dimensional mathematical configuration space, which still does not endow us with a measurable underlying physical mechanism that is less abstract and can be accessed through Reality.

Other accounts have appealed for the existence of extra dimensions, often associated with a particular topological structure that can allow for the entanglement to be maintained despite the spatial separation. According to one conjecture, the spacetime Reality lives on a 4D geometry that is the boundary of a 5D structure, held together by entanglement, which makes it the very thing that gives rise to spacetime (Van Raamsdonk, 2010; Cowen, 2015). Going even further, the $ER = EPR$ *conjecture* equates the EPR entanglement with the concept of *Einstein–Rosen bridge* (*ER bridge*, or *wormhole*, Einstein and Rosen, 1935) between black holes (Maldacena and Susskind, 2013). The black holes may even be microscopic in size and hence quantum (Susskind, 2016). In another model, a highly abstract graph-theoretic approach is used to unify quantum mechanics and gravity, in which nodes may be proximately connected in the graph space, despite being distant in the corresponding spatial metric (Markopoulou and Smolin, 2004).

In general, a higher-dimensional connection within the entangled subsystems, which can appear distant in local coordinates but arbitrarily short in hidden spatial coordinates, was considered to be the most viable avenue toward a solution of the nonlocality puzzle (Genovese and Gramegna, 2019).

8.2.5 Interim conclusion: The state of nonlocality in quantum theory

When the evidence of the various effects reviewed above is pooled, a reasonable conclusion is that nonlocality may manifest when local effects do not. In all cases, the effects are predicted from the formalism of quantum mechanics, which does not disclose any underlying, obvious mechanism through which the specific effects emerge, which leaves much room for interpretations and is not free from controversy. Therefore, we would like to propose a pragmatic take on the state of nonlocality in quantum mechanics[81], which can advance the present discussion: Nonlocality exists (cf. Bohm and Hiley, 1975, 1993; Popescu and Rohrlich, 1994).

8.3 Frequency as an ontic nonlocal dimension

In light of the mounting evidence for nonlocal effects in quantum mechanics, and despite several alternative explanations that have pushed back against the various nonlocal interpretations of these effects, we proceed to explore how an ontic frequency dimension may account for nonlocality. We begin by arguing for why frequency is inherently nonlocal. Then, we consider a Reality of 5D objects that comprise part local and part nonlocal elements. We proceed to apply Theorem 1 in the

[81] Presently in consensus with the mainstream opinion in the field, judging by the recent (2022) Nobel award given to John Clauser, Alain Aspect and Anton Zeilinger "*for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science.*" (`https://www.nobelprize.org/prizes/physics/2022/press-release/`; accessed 9.10.2024.)

general context of the three nonlocal quantum effects mentioned above and examine how frequency, as a nonlocal dimension, may provide a key to understanding them. Finally, we consider whether frequency should be thought of as a hidden variable, and whether the ontic existence of frequency may be sufficient to fully account for nonlocal effects.

8.3.1 Nonlocality of frequency

Nonlocality of frequency can be gathered from several of its properties. First, frequency cannot be reduced to either one of the two dimension types even with the additional assumption of determinism, despite the very close connection with both space and time (§3). Specifically, it was argued that frequency is dependent on physical parameters that exist in spacetime, but are not spacetime (§3.1.4). Second, the Fourier transform operation (Eq. 40)—the basis of much of the insight obtained about the relationship between time and frequency—requires integration over the entire domain. Therefore, to determine the temporal frequency (or spectrum) precisely, we need complete information about time (nonlocality in time), and equivalently, to obtain the spatial frequency we require all of space (nonlocality in space). The nonlocality in time and space can be somewhat mitigated through precise knowledge of the system's history, or through appropriate windowing (§3.5.6). Yet, it can never be completely eliminated due to the uncertainty principle that requires a finite window width for minimal spectral precision, as well as due to the compact support paradox that entails that either one or both the duration and the bandwidth are infinite. Third, these properties are also mirrored in the role that frequency has in perception: it is generally not perceived as a 'how often' experience, but rather as something completely different—a certain percept that somehow maps aspects of Reality that are detected by the different senses. The percept associated with frequencies, though, need not be localized: we can imagine or sense green, or G minor, or the texture of a strawberry without necessarily localizing it in space or time.

These nonlocal properties, though, do not relate directly to the question of whether frequency is an extra dimension or not. If the extra frequency dimension of P3 has any ontic validity, then in addition to these "standard" nonlocality properties that characterize frequency, it may also have the extra capability of delivering some of that spooky action at a distance, being neither in space nor in time.

8.3.2 Five-dimensional objects and entanglement

There may be more than one way to understand Reality according to Theorem 1. In the context of the nonlocality problem, we adhere to an interpretation that has Reality as 5D but the observer has a choice of whether to treat it as such (namely, as P3), or as lower dimensional, by setting different boundaries to the involved systems[82]. As a corollary, we propose to extend the idea of objects of perception from 3D to 5D, so that any physical object comprises three local spatial dimensions, one local time dimension, and one nonlocal frequency dimension. The nonlocality here is literal: it lies outside of space and time.

In the context of entanglement, the key point is to think of the entangled quantum system as an object—something that goes back to a comment made by Aspect (1999): "*We must conclude that an entangled EPR photon pair is a non-separable object; that is, it is impossible to assign individual local properties (local physical reality) to each photon. In some sense, both photons keep in contact through space and time.*" This statement echoes Schrödinger (1935), who wrote: "*Whenever one has a complete expectation-catalog—a maximum total knowledge—a Ψ–function—for two completely separated bodies, or, in better terms, for each of them singly, then one obviously has it also for the two bodies together, i.e., if one imagines that neither of them singly but rather the two of them together make up the object of interest, of our questions about the future.*" And finally, from the same paper: "*The whole is in a definite state, the parts taken individually are not.*"[83] To see how this relates to the frequency dimension, we refer again to the form of the two-particle entangled state of Eq. 75 and modify it to express a general parameter that can be used as a degree of freedom

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle_1 |B\rangle_2 \pm |B\rangle_1 |A\rangle_2 \right) \tag{80}$$

[82]The choice, however, may be constrained and therefore not all three modes are always available to choose from.

[83]It should be noted that not all entangled states necessarily entail nonlocality (e.g. Barrett, 2002; Méthot and Scarani, 2006; Brunner et al., 2014; Schmid et al., 2023). We focus here only on those entangled states that are nonlocal.

where the states A or B correspond to a discrete or continuous internal degree of freedom of the system that is used for differentiating the particles 1 and 2 (usually photons, but any particle combination would do)—spin, polarization, spatial momentum, orbital angular momentum, and time–energy (i.e., frequency). The standard notation of Eq. 80 is highly simplified, and thus, implicit to each particle state is its frequency dependence[84]. Regardless of the state of the particle A or B, each term of the entangled state, $|A\rangle_1 |B\rangle_2$ and $|B\rangle_1 |A\rangle_2$, has the same frequency dependence, so both terms on the right-hand side of Eq. 80 cannot be distinguished based on their frequency content alone. In more complex, multi-particle states, the allocation of frequencies to the entangled state terms is generally more complicated (e.g., Greenberger et al., 1990; Keller et al., 1998; Aoki et al., 2003; Shalm et al., 2013; Seshadri et al., 2022). For example, in a four-state frequency-bin entangled system, photon pairs are generated such that each photon can have one of two available frequencies, so that the photon pair has four possible frequency combinations (Seshadri et al., 2022). In this configuration each frequency component is found in two subsystems out of the four that make the entangled state.

8.3.3 Theorem 1 and nonlocal effects

Let us analyze the standard Bell test scenario according to Theorem 1. From the analysis of the measurement problem (§7), we know that the system is generally described as P1 before measurement and P2 afterward. Unlike the general case, though, there is an entanglement moment in the Bell test that is distinguished: when it takes place, the different subsystems momentarily become a single object. It means that it happens when time is dimensional—something that can be captured either within P2 or P3. However, as the particles that are being entangled "do not know" their future during that moment—they do not carry information about their measurement later in the experiment—this distinguished moment is necessarily indeterministic, which makes it a P3 moment and not P2. This is also the case because even though the entangled state still appears as P1 from the outside (or rather, it is modeled as such, given that a measurement has not taken place yet), the two subsystems are open systems to one another, within the scope of Theorem 1. Confirmation of this idea may be gathered from experiments by Moreva et al. (2014), who showed how an entangled pair of photons can appear to evolve in time internally, whereby one photon serves as a clock to the other (the "*Observer*" mode, corresponding to our P3), but when the two are measured as a single system from the outside, they appear static (the "*Super-observer*" mode, corresponding to our P1). Back to our analysis, the measurement finally triggers the instantaneous wave-function collapse across the system. It then causes the effective elimination of the frequency dimension as the system produces deterministic observation—belonging to P2—which no longer admits spooky action at a distance and thus appears local.

The remaining question is what evidence we have to justify the claim that the system internally behaves as though it is in P3, in which frequency is dimensional. Here are three relevant examples for the role of frequency out of several known secondary nonlocal effects that have been demonstrated for entangled systems and have no classical analogs. The first effect is *nonlocal dispersion cancellation*[85]. As a rule, pulses that propagate in dispersive media temporally broaden—their shape deforms. Most generally, dispersion can be approximated around the center frequency of the pulse ω_c (with corresponding k_c), with the spatial frequency depending on temporal frequency through $k(\omega) = k_c + \alpha(\omega-\omega_c) + \beta(\omega-\omega_c)^2 + \ldots$, where α and β are real parameters. Broadening goes as $\Delta\tau \approx 2\Delta\omega\beta d$, where $\Delta\omega$ is the bandwidth of the pulse and d is the distance traversed in the dispersive medium. When two photons are entangled, it can be shown that the dispersion broadening accumulated by

[84] For instance, in the context of *spontaneous parametric down conversion* (SPDC)—by far the most common method to generate entangled photon pairs—a single "*pump*" photon at frequency ω_p is nonlinearly converted to an entangled pair of "*signal*" and "*idler*" photons, related through conservation of energy with $\omega_p = \omega_s + \omega_i$, as well as conservation of momentum $\boldsymbol{k}_p = \boldsymbol{k}_s + \boldsymbol{k}_i$ (Burnham and Weinberg, 1970; Couteau, 2018). The simplified state can be expressed as $|\Psi\rangle = \frac{1}{\sqrt{2}}(|H_i\rangle |V_s\rangle + e^{i\phi} |V_i\rangle |H_s\rangle)$, where H is a horizontal polarization state, V is vertical polarization state of either the signal or the idler photons, and ϕ is a relative phase that can be modified (e.g., Kwiat et al., 1995). Shih (2003) derived an explicit corresponding state function for the similar state, $|\Psi\rangle = \frac{1}{\sqrt{2}}(|o_1\rangle |e_2\rangle + |e_1\rangle |o_2\rangle)$, which is defined according to the perpendicular (*ordinary*, signal) and parallel (*extraordinary*, idler) polarizations. The corresponding state function is then $|\Psi\rangle = \sum_{o,e} \delta(\omega_o + \omega_e - \omega_p)\delta(\boldsymbol{k}_o + \boldsymbol{k}_e - \boldsymbol{k}_p) a_o^\dagger(\omega(\boldsymbol{k}_o))\hat{o}\, a_e^\dagger(\omega(\boldsymbol{k}_e))\hat{e}\, |0\rangle$, where $a_o^\dagger(\omega(\boldsymbol{k}_o))$ and $a_e^\dagger(\omega(\boldsymbol{k}_e))$ are the creation operators of the two photons from the vacuum state $|0\rangle$, in the ordinary $\hat{o}$ and extraordinary $\hat{e}$ directions, respectively. The second delta function may be replaced with a more precise expression for a finite thickness of the nonlinear medium (Shih, 2003).

[85] The dispersion here relates to the *group velocity dispersion*, which is a second-order effect that deforms the envelope / group velocity of a finite-duration pulse and depends on the frequency square, in contradistinction to the linear dependence on frequency in regular (phase-velocity) dispersion.

one photon can be canceled out by applying the same amount of dispersion on the other photon, only with opposite sign of the dispersive coefficient β (Franson, 1992). A related nonlocal effect is of timing (time-of-arrival) correction using a dispersive medium for only one photon from an entangled pair, but without causing the classical broadening associated with dispersion, which gives rise to a fourth-order (intensity) interference (Steinberg et al., 1992a,b). The third effect is of sinusoidal frequency (or amplitude) modulation of one photon that can be observed on its entangled photon pair, so the combined modulations on the two add up nonlocally, either constructively or destructively, in the intensity interference pattern (Harris, 2008; Sensarn et al., 2009). All three effects involve continuous temporal and spectral changes to the photon spectrum and to the entangled state. While the mathematical analysis of all effects is generally done using Fourier transforms in a *deterministic* manner—just like in the radio example (§6.3)—the indeterminism inherent to the entanglement, the frequency dependence of all quantities, the distinction of the different time points, and the process that unfolds over time, renders this mode wholly P3, also much like in the radio example.

Theorem 1 indicates that there may be a discontinuity in switching between the three modes, which in the case of entanglement and nonlocality measurement should show as a discontinuous change between P3 to P2. Several Bell tests have estimated the speed in which the correlation seems to be nonlocally triggered / communicated between the two distant particles. In the most dramatic example, a lower bound in excess of 10000 times the speed of light has been reported for a loophole-free Bell test over 18 km separation between the entangled photon pairs (Salart et al., 2008). The general effect of discontinuity due to mode transition (§ 6.5), therefore, seems to be confirmed through the instantaneous correlation between the nonlocal parts of the entangled pair, as they revert to being strictly local and deterministic at measurement.

The other nonlocal dynamic effects we briefly reviewed above may be subjected to similar analysis, although somewhat more difficult to motivate because of the lack of an "obvious" nonlocal 5D object. Rather, we have to assume that all quantum objects, before measurement, can be internally described as P3. In the Aharonov–Bohm effect, the electron beams appear to experience a nonlocal interaction with a remote electric or magnetic field. The latter can be understood as causing modulation in the spatial frequency domain of the two electron beams (i.e., their momentum), with the lack of any other sources that may cause local modulation. (The electric effect would have an analogous modulation in the temporal frequency domain of the electrons.) Indeed, the original analysis of the effect admits a phase-modulation-like effect that enters the expected phase of the interference pattern (Aharonov and Bohm, 1959). Similarly, in the case of the two-slit experiment, the nonlocal phase difference that characterizes the interference pattern can be thought of as phase modulation that affects the frequency of the particle or wave, should it be taken to have complex amplitude. Although the absolute phase of the quantum state has no known physical role, if particles are 5D objects, they can nonlocally interact in the frequency dimension, where they coincide, but with respect to their relative phase. See discussion about a complex frequency dimension in §9.6.

8.3.4 Hidden local and nonlocal variable models

The nonlocal-frequency-dimension explanation applied to entanglement may get away with at least two paradoxical pitfalls, which may be more semantic than theoretical. First, frequency is not a hidden variable per se. In the state representation that explicitly contains the frequency dependence, it is not hidden—it is already part of the theory and is found in every state function (either as ω or k). It is also not a variable per se, but rather a dimension in P3, and a parameter in P1 and P2. Second, having frequency as an extra dimension does not contradict a Reality that is sometimes five dimensional. While highly unintuitive, Theorem 1, as well as the entire premise of this work, is geared to expand the concept of Reality in a way that is consistent with our psychophysical experience and brain structure, as well as with our mathematics, physics, and engineering practices. In this sense, Einstein's concern that spooky action at a distance violates Reality is warranted only if Reality cannot accommodate for spooky action at a distance.

8.3.5 What else is there to nonlocality?

Even if the hypothesis that frequency exists as a nonlocal dimension is correct, it is unlikely to fully account for the underlying mystery of nonlocality. The main difficulty is that even if the frequency dimension indeed exists and can provide the necessary link for the entangled particles, they may not be uniquely identified by it, or by the time–frequency combined coordinates (or time–bandwidth product; see, for example MacLean et al., 2018). Other systems such as additional pairs that are produced in the very same experiment may be identically tuned and have an overlapping time–frequency coordinates, perhaps with different instantaneous phase, and yet are not entangled to that pair. If the time–frequency coordinates of a nonlocally entangled system are its only identifier, then it would stand to reason that random pairs, identically tuned, somewhere in the universe, would get spontaneously entangled to the pair under test and they would be creating some disturbance to the observed pair. If this were true, then it would add an unknown amount of random noise to every Bell measurement, as well as to local entanglement studies, through the weakening of the entanglement of the system under test due to the "*monogamy of entanglement*" property (Coffman et al., 2000; Osborne and Verstraete, 2006). Therefore, there seems to be missing some kind of unique identification for the particles that can exist in a nonlocal space, which holds the information about the specific particle pair entanglement and distinguishes it from other, otherwise identical, particles that may coincide with the same nonlocal coordinates. We note that this shortcoming may not be unique to the present account of nonlocality. However, it is a tenet of quantum physics that particles of the same type are indistinguishable, so the suggestion that they can be uniquely identified is highly speculative. Alternatively, and perhaps even more bizarrely, the frequency dimension should remain private for the entangled pair.

Theoretically and in line with the logic of loophole-free Bell tests, we would always prefer to default to the local explanation and maintain a nonlocal one as a last resort. However, the nonlocality that a century of quantum mechanics has uncovered stems from the very essence of its theory: the quantum state, the divide between the stationary time evolution and the measurement operations, the uncertainty principle, and the associated lack of commutativity between conjugate operators.

For further discussion see §9.9.1.

9 Discussion

This work began with pointing to the central role of frequency in sensation. It was argued that the perception of frequency is typically abstracted from its standard ("how often") physical meaning, while typically being well-differentiated from the perception of both space and time. Contrasting the notion of extra dimensions in physics and in perception, we asked whether there is grounds for counting frequency as an extra dimension that is distinct from time and space. However, it was not possible to answer this question in a straightforward manner, because of the complicated, paradox-ridden, and even elusive interrelationship that frequency has with time: frequency is interwoven with time measurements, where it normally appears as a parameter that is the reciprocal of the period. In other contexts it appears as the reciprocal of the time variable itself, while yet in some other contexts it appears as a variable that is on equal footing with the time. We further scrutinized the possibility that frequency may serve as a dimension, by contrasting its properties with some of the key properties of the space and time dimensions of Reality. The analysis was encapsulated in Theorem 1, which counterintuitively tied together the dimensional status of time, frequency, and the status of determinism. The validity and applicability of the theorem was explored from wildly different perspectives in several examples that ranged from the simple epistemological application to the intricate and provocative ontological. The final example culminated in the suggestion that the nonlocal nature of frequency may be the underlying basis of quantum entanglement and nonlocality. The overall analysis, however, leaves much to be answered, and, undoubtedly, contested.

In this final section, we discuss several open questions that require further consideration and work.

9.1 Is frequency the correct quantity?

The entire premise of this work revolves around the possibility of temporal variation in frequency, which in itself is related to all other fundamental parameters or variables in oscillatory and wave motion. We argue that frequency is the most informative of them all, although it may not be obvious that this is indeed the case. Below are some arguments to favor frequency over other possible choices of related quantities.

9.1.1 Frequency and not period

The most basic definition of frequency (Eq. 1) describes how often an oscillation occurs and equates it with the inverse of the period T—the fixed duration of perfectly repetitive oscillation. Just as the instantaneous frequency (§ 3.5.7) is defined, there is no difficulty to talk about *instantaneous period* $T(t)$ that varies around the mean period and is the reciprocal of the instantaneous frequency. Conveniently, it is measured in time units and is easier to understand than instantaneous frequency. Inconveniently, it overlaps with the time axis and is often not well distinguished from time when taken to infinity (see §3.4.2 and §3.4.3). Computationally, it requires the reciprocal of the derivative of the phase

$$T(t) = \left(\frac{d\theta}{dt}\right)^{-1} \tag{81}$$

which is also unhandy. Frequency is somewhat more arcane of a concept to explain, though, being a reciprocal quantity, but easy to observe if thought of as a (real) number of cycles (or repetitions) per unit of time. It is also easier to relate to an arbitrary dimension—namely, any spatial dimension—where the meaning of frequency is retained, abstracted from the type of dimension at play (periodicity can apply to space too, but has a predominant semantic connotation of time). Importantly, dimensional frequency allows for negative values, which are conveniently continuous with the non-negative ones by virtue of the assumed validity of zero frequency. In contrast, the same cannot be achieved with the instantaneous period, which has to be continuous between $-\infty$ and $+\infty$ to correctly map the frequency axis, but has a singularity at 0. For all these reasons, dimensional frequency should be favored over dimensional period.

9.1.2 Temporal frequency and not spatial frequency

Wave dynamics is defined through the relationship between the spatial and temporal parts of the oscillatory motion. As was seen in §3.1.3, the wavenumber k is also the magnitude of the propagation vector of the wave, so the directional components are three spatial frequencies, k_x, k_y, and k_z, which may be independent from one another (Fig. 8). In this sense, a spatial frequency spectrum in which the direction varies dynamically in three-dimensional space would be three dimensional as well, whereas the temporal frequency spectrum is only one dimensional with the three spatial components projected on the time dimension. Therefore, the latter can be thought of as being more parsimonious, as well as geometry-agnostic. It also applies directly to oscillators that are not explicitly modeled with respect to spatial variations, and hence, it is more universal. However, it is important to remember that the two frequencies are related through the dispersion relations (expressed either as $\omega = \omega(k)$ or $k = k(\omega)$), so the information about the spectrum of one frequency type along with the dispersion formula of the medium should ideally be sufficient to derive the spectrum of the other (Fig. 17).

9.1.3 Frequency and not wavelength

The wavelength is the reciprocal of the wavenumber times 2π, which makes it the spatial analog to the period. Thus, the same arguments apply here as for the period (§9.1.1) and spatial frequency (§9.1.2).

9.1.4 Frequency and not energy

In quantum mechanics, the energy of a photon is related to its frequency through the proportionality factor that is the Planck constant h (Eq. 65). Additionally, all energy level transitions within quantized systems, involve either emission or absorption of photons, which correspond to sharp frequencies. Thus, in the context of a single photon with a sharp spectral line (constant frequency), the energy and the frequency contain the same information. However, in more complex systems energy can take different forms and can be transformed between them, so that this neat relation may no longer have any observational relevance. Furthermore, in many classical systems, there tends to be a *deterministic* relationship between the energy and frequency that is not as straightforward as the simple quantum one. Even when an explicit expression that ties frequency and energy is unavailable, it is possible to express the signal spectrum as an energy density function that can be integrated over a frequency interval to obtain the energy in a particular bandwidth (see § 3.5.6). Given all that, except for the simplest quantum systems, substituting energy for frequency in the dimensional context would only have an obfuscating effect on modeling.

9.1.5 Frequency and not phase

The most immediate definition of the instantaneous frequency is the derivative of the time-dependent phase function (Eq. 59). Therefore, the phase function can be obtained from the frequency function, up to a constant phase term. Moreover, when synchronized oscillations are modeled in nonlinear dynamics, the instantaneous phase serves as a generalized coordinate, along with its derivative, the instantaneous frequency (§3.5.9)—something that could be analogized to using displacement and velocity in standard Cartesian coordinates. This suggests that it is maybe the phase and not the frequency that should be considered the most informative quantity to serve as a dimension. Or rephrasing it in the negative—just as we would not prefer to have three dimensions of velocity instead of position, perhaps we should not prefer frequency over phase. Indeed, in the closest 5D theoretical model to the present work, Wiener and Struik (1928) opted for a fifth phase dimension, making more explicit the Klein (1926) 5D model that added a small spatial dimension x_0 that is curled by being inserted in the argument of a periodic exponent e^{ikx_0}.

However, there are several reasons for why using the phase is considerably less attractive than frequency. First and foremost, there are many situations in which the phase function cannot be directly measured, nor perceived. For example, at the light frequency range of the electromagnetic spectrum, incoherent imaging is the standard (also in vision), as only the intensity can be detected and not the amplitude or phase of the light waves. Similarly, the *power spectrum model* of hearing, accounts for the insensitivity of the ear to phase changes with many typical stimuli (see references in Weisser, 2021, pp. 113–114). It has been successfully employed in numerous audio applications, which rely on a frame-based, time-windowed computation of the acoustic signal and are not concerned with its instantaneous phase or frequency—only with moving-average kind of quantities, within each frame (see §3.5.6). A similar way to state the same thing is that there are many situations in which we do not care about the exact part of the period where the oscillating system is positioned, but rather, how often it oscillates on average (Eq. 61), which is much easier to estimate. Yet another way to restate it is that for signals that are stochastic there is no valid phase function, and yet their frequency can be validly estimated statistically (§3.5.2–§3.5.5). In realistic broadband signals, the situation can be even more complex, because there may be no unique way to represent the phase of the different carriers, unless they are well separated in well-defined narrowband channel boundaries (Boashash, 1992).

Another reason to disfavor the instantaneous phase is that even if it does exist, it is constantly changing, even at times when frequency is, for every intent and purpose, constant. Therefore, frequency is more parsimonious, as it expresses the same thing using a single number (in the quintessential case of a constant frequency) rather than a function.

Finally, following from the previous two reasons, the instantaneous phase function carries little unique information, because the (wrapped) phase values are bounded on a 2π interval. Therefore, a random sample of the phase function would not give any orientation for the frequency range in which the system oscillates without further calculations. In contrast, the frequency range is unique and provides a much more informative description of physical phenomena, which matches perception, and in general, a slower observation of the physics at hand.

9.2 Challenges to the inclusion of frequency as its own dimension

9.2.1 Role of frequency in partial differential equations

Partial differential equations in physics distil the complete dynamical account of the systems they model. Only few of the simplest such equations were mentioned in the section about wave motion (§3.1.3). In many of those equations, frequency never appears explicitly in any of the parameters, but may instead be indirectly parametrizing another parameter, giving rise to dispersion, or to another form of frequency dependence that is implicit to the problem. Only few equations explicitly model spectral dependence as an additional variable, whose derivatives with respect to time and space appear in the equation[86]. However, unlike the other dimensions, derivatives with respect to frequency itself do not generally appear in those equations for the complete description of any system. What is the point, therefore, of adding frequency dependence to an already complete description?

To answer this, we have to backtrack briefly and understand what such partial differential equations assume. The entire premise of these equations is to provide a complete description of well-posed

[86]The notable exception are nonlinear differential equations (usually ordinary rather than partial) that model the complex dynamics of synchronized oscillations, which rely on instantaneous phase and frequency functions (see §3.5.9).

problems, where the system is by definition isolated from the rest of the universe (P1) or the system is the universe itself for the sake of the problem (P2). Those parts of the universe that are considered pertinent to the dynamics of the system should be recognized and appear as additional terms in the equation. Nothing else but them exists in the problem formulation, which by its very definition does not have room for unknown unknowns. In contrast, for frequency to be its own dimension, the system must become open (P3). Its open boundaries, if they are at all defined, may or may not be breached by external forces whose magnitude, direction, and duration are unknown and cannot be predicted with certainty, although they may be tracked. These types of problems with uncertain inputs and outputs often require the help of signal processing techniques that are not committed to a specific problem configuration and are more generally capable of dealing with real-time processes, often with the aid of a statistical baseline around which local variations are quantified. Hence, it is impossible to express in closed-form the lack of knowledge while retaining a *deterministic* dynamics of the equation. Rather, these uncontrolled inputs, inasmuch as they exist in observation, may be regarded as unwanted noise and uncertainty. If large enough, it would cause for a measurement to fail. An example for this idea was given in §6.2.

9.2.2 Possible conflation of different types of frequencies

When translated back to the perceptual domain, an extra-dimensional physical reality may lead to conflation between frequencies whose source is macroscopic (e.g., touch and sound) and frequencies whose ultimate source is electromagnetic (e.g., vision, electroreception in fish, infrared thermoreception in snakes and lizards). The macroscopic quantities represent vibrations or other mechanical oscillations, whereas the electromagnetic frequencies stem from quantum processes at the subatomic or molecular level. It is not clear that the frequency dimension universally converges between these senses or others, although recent evidence suggests that the low-frequency sound and high-frequency tactile vibrations—both below 1000 Hz—are processed by the same cells in the inferior colliculus of the midbrain (Huey et al., 2025). More generally, however, although the low modulation-band frequencies of different modalities may become comparable after demodulating the sensory input, they are generally not perceived as equivalent once in the different modality pathways (perhaps with the exception of special cases of synesthesia or a strong sensory binding of stimuli emanating from a common source).

9.2.3 Multiple frequency dimensions

Another challenge is that the frequency dimension may not be singular. Physical objects have multiple frequencies going on simultaneously, which can be understood as independent degrees of freedom—some representing oscillations or vibrations while others map to rotations. For example, a rigid body has three degrees of rotational freedom—each of which can be taken as one frequency dimension corresponding to one axis. One option is that in perception, all of these frequencies are projected on a single dimension, so that the union of all dimensional frequencies is perceived as the spectrum of a single object. More complicated options may include more than a single frequency dimension in perception either in all or in a subset of the available modalities. Or alternatively, in some cases independent low-frequency information modulates high-frequency carriers and becomes its own dimension after demodulation (for example, see arguments for a two-dimensional spectrum in hearing; Weisser, 2021, p. 123–125). See §9.3.

9.2.4 Trivial addition

In a sense, the addition of frequency as a mandatory dimension to space and time is trivial. For example, the universe is mapped through observations at different electromagnetic wavelengths, as there are dedicated instruments for radio, microwave, infrared, light, ultraviolet, x-ray, and gamma radiation astronomy. While the objects on the resultant maps are taken as 3D projections, their existence is only revealed if observed in the appropriate spectral (and temporal) windows. If observed in the wrong wavelength, they are effectively invisible. This is not all that different from the sensory inputs to perception. Objects can be completely invisible (either dark or transparent), unless they produce or reflect light in the visual range, sound in the audible range, etc.[87]. Any notion of

[87] However, regardless of their far-field detectability, solid objects seem to be always detectable by touch, as long as they are large enough to actuate some of the mechanoreceptors in the skin, or to forcefully act on the body. Even here, though, a totally static tactile stimulus—one that never changes (i.e., zero relative velocity between the object and perceiver) can be modeled as

triviality, though, reflects a parametric approach to frequency, in which time-invariant filtering is sufficient to extract average, coarse spectral information about the object, which is ultimately static in nature. Instead, we emphasized the time-varying nature of spectral information whenever real-time applications require it, as is most obviously the case in perception.

9.3 Nested dimensionality

It was mentioned several times along the analysis that there is an association of low-frequency modulations with the emergence of a frequency dimension that is locally analyzed, within a certain time window and spectral bandwidth. Within this time–frequency locus, however, all the limitations of frequency analysis still apply—primarily the uncertainty principle (§3.5.6). So, for example, we may be able to demodulate a signal and precisely eliminate a high-frequency carrier from it, in which case we are still left with a low-frequency baseband, whose frequency content may be determined using more or less precise analytical tools. Given that there is no mathematical limit to how many layers of signal modulation are possible, it is possible to characterize further demodulated signals using the same three modes, in which no-time, determinism, and frequency have a mutually exclusive relationship. A commonplace example is speech, which contains audio-range carriers at frequencies above 100 Hz that are naturally modulated at lower frequencies, typically peaking at around 3–4 Hz (Steeneken and Houtgast, 1983; Drullman et al., 1994). Further nesting is possible if we let such naturally modulated audio signals modulate radio carriers of frequencies in the hundreds of kilohertz and higher. Incidentally, the sense of hearing itself is sensitive to both carrier- and modulation-domain frequencies[88] and information from both is extracted continuously when listening to speech, music, or any other sound. Therefore, hearing appears to constitute an organic example of a nested-dimensionality system. Whether nesting is strictly mathematical and epistemic or has additional ontological correspondence remains an open question.

9.4 Implications on time as a dimension

Despite the importance of both time and frequency in the harmonic analyses of numerous physical systems, only time has been nominated as an own dimension (although not without dissent; e.g., Barbour, 1999; Rovelli, 2019). Moreover, to my best knowledge, in the various musings about time, frequency has never been considered a relevant concept in the discussion, beyond being a necessary component for building clocks—the essential measurement tool to estimate the passage of time (Audoin and Guinot, 2001). This implicitly assumes that the concept of frequency is not plagued by similarly crippling paradoxes. The applied mathematical and engineering literature, however, suggests otherwise (Boashash, 1992).

The current analysis still gives primacy to time that is distinct from periodicity as the first intangible dimension to emerge out of a 3D spatial reality, yet it demonopolizes it from being the only one (§4.1.8). Because the definition of frequency is intertwined with periodicity, time, and many other specific physical parameters, it may appear to be somewhat less mysterious than time, and perhaps more amenable to a straightforward definition. However, the mathematical nature of the definitions of instantaneous frequency, as well as their highly indirect nature that requires some knowledge of physics in order to explain them, seem to make it no less abstract than time. The complicated story behind frequency—its earliest definitions, slow mathematical evolution in establishing the proficiency for working with it, the late development of physical filters and the theory behind them—may render it more suspect than time in its primacy. And yet, this intertwinedness with time may be seen as part of Reality that only compounds the underlying mystery of it, rather than alleviates it.

Theorem 1 formalizes the conditions for which time can be elevated to become (or emerge as) its own dimension, rather than a parameter. This may add some weight on the age-long controversy about the nature of time and its role in Reality—whether it is a real property of the universe, or an emergent one that only serves our perception of it. The ideas and results that were derived here by following the strict definitions and application of frequency that concluded in realities of type P1 and P2 are not unlike those that were reached at using much more elaborate physics (Rovelli, 2019). The convergence of these markedly different approaches gives credence to the radical and perhaps

being exactly 0 Hz in its Fourier frequency sense—is arguably not going to be registered by perception as a valid stimulus (think of the floor under your feet when seated motionless for a very long time, before you read those very words).

[88] Some authors referred to the modulation spectrum as the periodicity domain of hearing, and to the corresponding mapping as *periodopy* to distinguish it from high frequencies that are perceived as tonal and are mapped by tonotopy (Langner, 2015).

unintuitive notion that time is an emergent dimension of Reality, rather than a fundamental one. Nevertheless, even in P1, time appears as a manifestation of periodicity, which we understand as a parametric incarnation of time, only not one that produces events that can be reliably distinguished from the whole, given that the notion of past, present, and future appears void. Therefore, saying that time altogether does not exist may be a stretch, or a definitional or logical matter (McTaggart, 1927 / 1993). Instead, the notion that time may be fundamental to Reality (Smolin, 2019)—albeit a changing notion of time as the modes change—may be more in line with the present work.

See §B for further metaphysical connections between Theorem 1 and time.

9.5 Boundaries

The formulation of Theorem 1 is implicitly reliant on a particular configuration of the boundaries around physical systems, where it was argued throughout that different choices on how to bound the system can lead to different dimensionality in the physics itself—whether only apparent or real. It begs the question, though: how are boundaries set? And, is there complete freedom to arbitrarily position the boundaries in space around what we consider as being "the system"? Why are some configurations justified and others are not? In most analytical problems, the choice is more or less inferred from the problem definition, where a person (the scientist) sets the problem and decides what is pertinent to its solution and where the mathematics and geometry can be facilitated or approximated while maintaining rigor and convergence with relevant empirical findings. An insightful remark by Boltzmann (1974 / 1899, pp. 119–120) aptly describes the situation we face upon deciding what the appropriate boundaries are for a given system we are interested in: "*We must therefore include the whole Earth as part of the surroundings of a gravitating body, but leave the Moon and stars out of account, since they have no noticeable influence. It is thus once again a pure assumption, to be subsequently justified by experience, that we can always draw the boundaries of immediate surroundings in such a way as to include all essentials, and thus actually arrive at a formulation of laws of motion.*" However, in Reality that is not actively modeled by anybody, it is not at all clear that any of this reasoning has any bearing whatsoever. For example, if a leaf falls on a rock, neither does the leaf nor the rock should "care" about maintaining their boundary, which—in our human eyes—is likely to be visually obvious and robustly maintained for a long while after impact.

In the discussion about the measurement problem of quantum mechanics, the boundary-setting problem is a common thread, as different interpreters (including the present account, should it be mistaken for one) sometimes have different takes about the closedness of the quantum system before and after the measurement (§7). In the present account, this extended dramatically to the realm of quantum nonlocality as well, where only the change of boundaries could disrupt the apparent nonlocal effects that make a 5D object whole (§8). Actively maintained boundaries seem to be a biological feature across all domains of life, where membranes are the fundamental structures that spatially and temporally regulate what is inside and outside to the organism (Watson, 2015). Although much of it may be achieved automatically on a biomolecular and biophysical level, it is arguable whether, as lifeforms get more complex, it is possible (or even conceptually coherent) to maintain boundaries without agency. Does a tree maintain its boundaries actively or the tree simply is—passively being molded and breached throughout its life? Or, even if taken to an extreme—suppose we have a high-power computer program that is connected to mechanical hardware that allows it (the computer + software + auxiliary hardware) to maintain its boundaries, expand, or contract, so it is automated on some level. This program still had to have been initiated by some agency, if only through a very primitive program or goal to set it off. Otherwise, it is hard to make sense of the notion that the computer program genuinely "cares" about its boundaries in any sense that we can relate to as conscious beings, whose very lives depend on the properties of its membranes—both real and imagined[89,90].

A final comment would be in place regarding boundaries that define objects. In classical optics and mechanics, objects are defined geometrically, as three-dimensional entities that have some

[89] For a dramatic example, this video shows the death of a photophobic protozoa of genus *Blepharisma*, whose membrane is ruptured due to light exposure: https://www.youtube.com/watch?v=4bj6SqgT4SQ ("*Single-celled Organism Dies*" by James Weiss, uploaded on 25.12.2018, accessed 30.12.2024).

[90] I have not been able to find a discussion in the literature about boundaries that specifically considers the essential role of agency in defining them. However, a quote from Mermin (2012) in the context of QBism as an interpretation for quantum theory (§7.6) comes close to what I expounded on above: "*Shiftiness, vagueness, and ambiguity all arise from a failure to realize that like probabilities, like quantum states, like experience itself, the split belongs to an agent. All of them have their own split.*"

structure related to their in shape, composition, dynamics, etc. As long as time is parametric only (P1), the object simply vibrates and moves, but it may not come in and out of existence. If time is a dimension too, though, then the object boundaries can be defined temporally as well, which can then include the entire history of the object in space. Given Slepian's compact support paradox (§3.4.5), we know that a finite object in time—one that can be defined using a finite time support function—would necessarily have a spectrum with infinite support, which does not meet an intuitive criterion of a physical object that is finite in all dimensions. Therefore, it is consistent that P2 best entails four-dimensional objects, but not five dimensional, so parametric frequency is omitted from the object description. Finally, if we want to refer to five-dimensional objects notwithstanding, then we can take it in P3, but then we lose the rigid spatial boundaries from before: we get to have a 5D object that is local, but we do not control its spatial extent, because the system referred to is open and need not conserve energy, matter, or anything else. This simple conceptual analysis that spells the inherent relations between dimensionality and boundaries according to Theorem 1 contains the seed of nonlocality that was argued for in §8.

9.6 Complex amplitudes

One aspect of the proposed frequency dimension has been only mentioned in passing and was largely left out of the discussion throughout the text: the Fourier spectrum that characterizes both P2 and P3 requires two numbers per frequency point: amplitude and phase. The phase cancels out in power spectral estimates as are employed in P1, but it is essential to accurately reconstruct some other types of spectra. There are different ways to express this duality—the most common of which uses the complex exponential, which has a real frequency, but can take a complex amplitude that incorporates the phase. The two-dimensional complex plane is then used in full to express the independence between two orthogonal components that can be used to reach an alternative description: the in-phase and quadrature terms (the orthogonal sine and cosine solutions; Footnote 62)[91]. The latter formulation is sometimes used in radio communication and classical physics and electronics problems, but is considerably less handy to employ as a substitute for the complex exponential of the Fourier transform that is defined over the complex plane.

In other words, it seems that the extra frequency dimension is perhaps better thought of either as a single complex dimension or two real dimensions. Both possibilities do not increase the sense of intuition of the already unintuitive nature of frequency as an extra dimension.

Nevertheless, there may be some justification in opting for a complex frequency dimension. Standard quantum theory is axiomatically formulated using states and operators that produce real observables, but are defined in the complex rather than the real Hilbert space—something that has been repeatedly challenged (e.g. Stueckelberg, 1960; Caves et al., 2002; Aleksandrova et al., 2013). Unlike classical physics, the use of complex numbers in quantum mechanics appears to be more deeply ingrained in its formalism. Alternative real-number quantum theoretical formulations contain double the number of real Hilbert-space dimensions than for complex Hilbert spaces (of finite dimensionality), which generally leads to identical predictions as the standard complex formulation, with the addition of relatively few constraints to make the two domains match. However, a recent work by Renou et al. (2021) proposed a benchmark Bell-like test and inequality, which involve three particles in an *entanglement swapping* experiment (Zukowski et al., 1993) that can distinguish between real and complex predictions. A subsequent test indeed established that a complex formulation is necessary to predict the results (Chen et al., 2022). While this may not be a result that can be generalized to all of quantum mechanics, it lends credence to the standard complex formulation.

The paradoxical quantum-state formulation on the complex plane may be readily accounted for if we accept the frequency to be dimensional and its amplitude complex, which would be then directly imparted to all quantum states, pre-measurement. This speculative idea would require further exploration to find out if it has any merit.

9.7 P2 = P1 + P3?

All three modes are special and each brings something else to the fore that the other two may not be good at. Of the three, P2 stands out as the only mode that is deterministic. Incidentally, it is also the one mode that best corresponds to classical physics and its logic.

[91] Note that the phase and amplitude are not independent, in general, unlike the in-phase and quadrature components of the complex signal (Couch II, 2013, pp. 450–454 and 461–463).

Between time, frequency, and determinism, the latter may be the odd one out, not being a strictly physical parameter that can be straightforwardly quantified[92]. It has two opposing facets. Internally, it represents a judgment of complete knowledge—we must determine whether a system is truly closed, whether we have characterized it to a sufficient degree, and whether our estimation of its behavior is precise and unbiased. Without these affirmations, we cannot deem that what we know about the system is deterministic. Alternatively, we can decide—by fiat—that we are only concerned with the present degree of determinedness that we can achieve—a certain degree of precision and complexity that we can muster—and work it analytically from there under the assumption of determinism. In both cases, we are going to learn something about Reality. In either case, we are left with the metaphysical question of determinism: Do we know that nothing is ever going to change our assumptions that had led to the determination of determinism? This is the age-long *problem of induction* in a different guise (Hume, 1740, 1748). How do we know that the physical laws of today are going to be the same physical laws of tomorrow? Or that the probability distributions are what they seem to be? Or that the universe is finite and closed? All these heavy questions may never be answerable. Or maybe they can be answerable for a brief moment of confidence, given the state-of-the-art in both data and theory. In a physical landscape in which every moment appears identical to every other moment, the experience from one brief moment is readily generalized to others, producing ad-hoc patterns and rules that are only correct as long as the generalization is warranted.

This somewhat circular set of considerations may lead us to the formation of an epistemological rule of thumb. As observers, we can only aspire to bring our reality to an ideal P2 Reality, in which everything is known with certainty at arbitrary precision. But we are forever constrained by what we know and what we are able to know. And yet, by virtue of acquired knowledge and memory, whatever we acquire from the various patterns and probabilities we observe in P1, which are modulated by the present course of events that we track in P3, enables us to approximate P2 that is composed of these two. The indeterminism of P1 relates to the details and to the specifics of individual records, whereas the indeterminism of P3 lets us determine the proximate time windows precisely, but is blind to the overall, longer-term trends (§5.3). Whatever is left out may remain forever underdetermined. The difference between our deterministic image of Reality (i.e., reality) and Reality itself is whatever is left uncharted by P1 and P3 and would variably show as errors, uncertainty, or more technically, noise. These are anyway inevitable when dealing with physical quantities represented by real numbers, which can never be known or represented at infinite precision, as is required by determinism (Del Santo and Gisin, 2019). Thus, we summarize this epistemological rule with

$$P1 + P3 = P2 + noise \tag{82}$$

9.8 Theorem ontology and epistemology revisited

There is a way to interpret the theorem that was not pursued in the examples, but which may be valid, albeit unprovable (§5.1). It is possible that while the epistemic usage of the various modes is correct, Reality is veridically constrained to one mode only. Such a view—or maybe, a belief—should resort to these strong statements as are implied by the modes all the time—something that is not entirely unfamiliar in ongoing metaphysical discussions. So, for example, a belief that the universe is deterministic down to the most minute movement would entail that it is indeed a P2 Reality. Or, beliefs that everything is random, or that it is all about luck, are indicative of P1. Or, maybe things repeat periodically as in P1, according to one reading of Ecclesiastes 1:9: "*The thing that hath been, it is that which shall be; and that which is done is that which shall be done: and there is no new thing under the sun.*" Or, time does not flow—it would also be P1—in the words of Parmenides (Fragment 8; Burnet, 1920): "*Nor was it ever, nor will it be; for now it is, all at once, a continuous one.*" This is contrasted with Heracleitean / Platonic "*all is motion and flux*[93]" would have us at P3. A final and more recent example subscribes to determinism that is backed by statistics, so indeterminism here is nothing but randomly distributed life variables, forming a bleak (or, perhaps, sober) combination of both P1 and P2 (Sapolsky, 2023).

In a similar vein, some may be beholden to one of the competing propositions that were logically rejected in the proof of the theorem (§5). For example, P4 states that both time and frequency

[92] I am quite certain that there is never going to be a "determinism meter."

[93] This famous paraphrase of Heraclitus's doctrine appeared in Plato's dialogs Theaetetus and similarly in Cratylus, although the closest aphorism remaining from Heraclitus may have be in his Fragment 12: "*As they step into the same rivers, different and (still) different waters flow upon them.*" (Heraclitus, 1991).

are independent dimensions, but the universe is deterministic nonetheless. So here, some may claim that it is maybe only our inability to access this very real determinism, but we instead experience reality as chaos unfolding in different degrees, due to our own limitations as humans.

Another point of departure can be the notion that the universe had a definitive starting point, such as the Big Bang, which was preceded by nothing, so time did not exist before. Therefore, time must exist, but may not have to end, if the universe will keep on expanding indefinitely. Time is dimensional here so P1 can never be true.

To be able to hold on to any one and only one of these beliefs requires that contradictory views (represented by at least one of the modes or their tendency to switch) should necessarily be void. Any useful observational data or theoretical advent that is done using these modes has to be a form of extreme idealization that is only a means to an (epistemological) end.

While these beliefs may have variably played significant parts in philosophy, metaphysics, and theology, they may not be directly amenable to the kind of logical analysis that has been pursued here. We can only submit the idea that Reality (or reality) is instead these three modes together, but not simultaneously. On the metaphysical front, this strange view may receive some currency, by lending an unexpected solution to at least one persistent metaphysical problem that is associated with such unimodal worldview as was alluded to in the above paragraph—the so-called foreknowledge problem that relates to free will. This is developed separately in §B.4.

9.9 Dimensional frequency and physics

The possibility of frequency constituting an extra dimension of Reality may be at odds with several ideas in modern physics that were not dealt with explicitly in this work.

9.9.1 Ontic frequencies, nonlocality, and relativity

The farthest that the present theory ventured has been to associate the frequency dimension with quantum nonlocality and to suggest that the frequency dimension can be, quite literally, neither in space nor in time. We suggested that there should be a meaningful notion of 5D objects that are defined by virtue of dimensional frequency, which can form a connection over the spatial divide between remote entangled particles that form the object and are characterized by a joint (time-dependent) spectrum (§8.3.2). While this idea may seem unrealistic and nonphysical on its face, it can also be seen as a logical consequence of taking the already strange effect of action at a distance and combining it with dimensional frequency—a nonlocal quantity in its own right.

Interestingly, the logic that has led us to associate frequency with quantum nonlocality was obtained with no direct reliance on quantum theory, but instead on logically contrasting the basic definitions of time, frequency, and determinism that are applicable universally. The same rationale would therefore lead us to the uncomfortable conclusion that frequency can be nonlocal regardless of scale. And yet, most of physics appears to strictly act and interact locally. A conservative, albeit opaque, prescription may be framed in the negative: nonlocality is what is left from Reality after subtracting all local effects.

One difficulty associated with the notion of nonlocality and frequency may be a conceptual inability to completely separate locality from nonlocality. Presently, the longest nonlocal correlation that has been reported was over a 248-km distance (Neumann et al., 2022). Does this distance have a limit? Whether the entanglement information goes through spacetime superluminally or is outside of it, it may breach the territory of impossibility according to the theory of relativity.

The question of distance is left unanswered also in the cases of the other two nonlocal effects reviewed—effects that are not usually grouped together with entanglement—the two-slit experiment and the Aharonov–Bohm experiment. If, hypothetically, the geometry of these experiments could be scaled to include distances that are relativistically relevant, would the nonlocality hold? Would it carry all the way to infinity? Cosmologically-inspired conjectures combining gravitational and quantum field considerations generally seem to assume that nonlocality can indeed span arbitrary distances in the local universe (e.g., Maldacena and Susskind, 2013), but without empirical evidence we remain agnostic about it here. In all cases, the relativistic meaninglessness of absolute spacetime coordinates (§4.1.1) challenges the notion that quantum nonlocality should be associated with frequency. Our idea was that 5D objects can be defined nonlocally, which implies that their spectral "coordinates" should hold at arbitrary distances. However, this suggests an absolute rather than relativistic value of frequency (or even a range of frequencies), which does not sit well with relativity theory. An absolute role of frequency values was highlighted already in §4.2, as making the frequency dimension unique compared to spacetime, and giving rise to the specific world we live in, which is not spectrally-invariant in a large number of contexts. It was already inferred earlier that frequency is unlikely to be the full explanation behind remote entanglement (§8.3.5). This missing extra component that is needed to identify entangled particles is perhaps also capable of resolving the inconsistency with relativity that is noted here.

9.9.2 Relationship with other models of extra dimensions

As was mentioned in §2.2, the possibility of extending physics to include additional dimensions to Reality has been thrown around for more than a century, although previously the focus has been on spatial dimensions, which are unmeasurable, at least at present. Some of these efforts were guided by mathematical issues and inconsistencies in the standard formulation of relativity, quantum, and classical physics. These aspects make these models fundamentally different from the present take on dimensional Reality, which was motivated by reasons pertaining to perception and to some puzzling issues regarding the concept of frequency. This exploration revealed a paradoxical structure of Reality with a variable number of dimensions according to Theorem 1, which does not compare straightforwardly with any other higher-dimensional physical theory. The addition of frequency as a dimension does not directly say anything about the number of spatial dimensions in the universe. We only used the spatial dimensions as a scaffolding upon which to have wave propagation, and to be able to meaningfully speak about dispersion—the variation of the wave speed in the medium as a function of frequency. Dispersion was used in the negation of propositions P5 and P6 in §5, which means that we made a hidden assumption that the universe contains at the very least one dimension of space. The inverse should also be true: frequency is a key quantity in all physical theories regardless of how many extra spatial dimensions they may have, so the present theory may not threaten their hypothetical validity. That said, the variability in the number of dimensions that is entailed by the theorem is in itself at odds with all other theories. Still, there may be a tacit convergence between them in that even higher-dimensional models should ultimately be observed or perceived as four dimensional, which is P2 in our case—the end point of most measurements and analyses (see §9.7).

9.9.3 The impending "doom" of spacetime

The present work has taken a diametrically opposite approach to contemporaneous attempts of resolving the discrepancy between perception and physics, as well as overcoming inconsistencies within physics proper (Hoffman, 2024). State-of-the-art theoretical science as is narrated by Hoffman and colleagues seems to be going in the direction of harnessing ever more sophisticated mathematics, which tends to be both arcane and abstract. It offers the necessary freedom to create what are deemed as more fundamental mathematical structures, from which spacetime can be derived as a secondary outcome, whereas concepts such as information, probabilities, entropy, and even evolutionary mechanisms for survival of life are considered primary and primitive.

Rather than abandoning space and time as primitive concepts or structures that are a byproduct of our perception having to interface with Reality, we treated them (at least space) as a given. We sought to resolve inconsistencies that had not been previously associated with any major paradoxes in physics or philosophy, but, at most, only as a footnote in time–frequency analysis research. Not only did we not introduce any new mathematics, but the mathematics we appealed to in order to support our argumentation is relatively basic and maybe even disappointingly so, in light of the promising possibilities opened up by modern mathematical and theoretical physics. Instead, we relied on the logical consistency of conditions and contexts in which the quantities are used. We also sought to bring together practices from different fields and connect the dots of evolving science over four centuries, which have not been properly united previously. Critically, we also treated sensation and perception as epistemologically instrumental for the understanding of the physical world—something that also defies the current trend in research, which tends to highlight the illusory, imperfect, distorted, cognitive, and heavily processed nature of perceived information in all modalities, which is otherwise inferior to instrumental and scientifically grounded epistemology.

One way to interpret the discord between the present and Hoffman's approaches is that the present theory of frequency is nothing but a reframing of subjective reality that may match objective Reality more universally than is entailed by the classical definitions. Here, our approach is still classical at heart and, at best, improves on the current inconsistencies between perception and physics, but ultimately fails to provide a route for resolving more pressing issues in the science.

Metaphysically, this interpretation can imply that perception is fundamentally hopeless: it is a tool that is subservient to the biology and any resemblance between it and the external Reality is either accidental or utilitarian, but not principal. It would be thusly understandable should the present work, which makes what may seem as extraordinary claims about well-trodden concepts in the science, may even come across as an anachronistic step back from where the state of the art lies.

An alternative way to interpret the present work is to see it as providing reinforcement of several otherwise shaky connections between reality and Reality. These may have been previously neglected because of a presumed historical split in the sciences (§2.3), focus on other issues, or prevalent beliefs about the limited capacity of human perception to tap into Reality, most poignantly expressed by Kant (see Footnote 1). This interpretation hints that science can do better to bridge the seemingly insurmountable gap between the inner and the outer—the subjective and the objective—and provide a more comprehensive outlook on how the two relate to one another, before resorting to the total abandonment of space and time. Taken to the extreme, this perspective may even suggest that where reality and Reality meet they have to be one, somehow—as a sort of a condition on their continuity being of the same world. In contrast, the prevalent situation in most sciences is that the two are disjoint and the subjective is artificially and arbitrarily introduced along the way, by the experimenter, analyst, engineer, or theoretician—all depending on context, so it tends to hide behind different guises: observation, agency, decision, choice, volition, criterion, goal, threshold, preference, program, etc. Arguably, this has been a very useful relic of Cartesian dualism that has the mechanics of life completely detached from its nonphysical component—be it consciousness, mind, thought, spirit, or soul (Descartes, 1637 / 2004). The separation of the physical and the nonphysical is spectacularly effective far away from the points of interface. But, at the physical–nonphysical interface it either stops working or gives rise to paradoxes of different kind, which are all the more accentuated in the realm of psychophysics. Whether the addition of frequency and the elusive dimensional nature of both time and frequency implied by Theorem 1—what spells out an odd bridge between the internal reality and external Reality—can do something to alleviate the sense of doom that has been noted of late, remains to be seen.

9.10 The three modes of reality and the two brain hemispheres

During the analysis of all of the examples in §6–§8, we encountered what may be taken to be key points that require agency. They manifest either in the choice of analysis, which can be considered epistemically necessary, or they are implicit to the process that is needed to achieve a particular task, not necessarily by way of explicit analysis. In both versions, it may naturally pivot the discussion to the role of the brain in producing perspectives on Reality. It was additionally suggested that consciousness has to be involved in the creation of boundaries, which has meaning only when there is something or somebody to draw or detect them, since they are rarely, if ever, self-evident (§9.5). Although it is—to many—a very unsatisfactory interpretation of the quantum measurement problem, it is not clear how it is possible to altogether remove consciousness that sets up a measurement, observes it, and analyzes it, no matter how much removed it is from a long chain of such events. Putting all this aside, though, we provided what seems like a completely unrelated example from psycholinguistics, wherein we identified similar patterns that are hypothesized as models for intricate, routine tasks of the brain—the disambiguation of word meanings in running speech (§ 6.4). The meaning disambiguation models can be thought of as an epistemic reflection of the brain processes, or reflecting the epistemology of the scientists that originally formed these models. In any case, at least in that one example, it was possible to identify a pattern that is inherent to our thought process—whether conscious or unconscious (automatic)—that converges with the modes of Reality as are spelled by Theorem 1. In fact, a theory by McGilchrist (2009) that accounts for the grossly different roles of the two brain hemispheres seems to be concordant with a rough classification into two of the three Reality modes that we find in all of our examples.

McGilchrist (2009) attempted to provide a reasoning for why the brain needs two hemispheres that are nearly identical anatomically and physiologically, given that the functional *lateralization* (i.e., more prominent brain action either in the left or right hemispheres) has been a long-time unsolved puzzle. Using countless examples from neurology, neuroscience, psychiatry, psychology, linguistics (and subsequently also from the arts, philosophy, sociology, and history), McGilchrist makes the case that, roughly, the left hemisphere is responsible for generalization, pattern finding, rule formation, habit formation, and abstraction—all the things that require sustained observation over long time until a nearly automatic, impersonal, time-less behavior can be formed and applied.

In contradistinction, the right hemisphere reacts more to the present, to things (sensory data) as they are in their own context, without necessarily naming or classifying them. The right hemisphere does not engage in abstraction, so it is more attuned to novelty and to the uniqueness of the moment. The two hemispheres may react to the same Reality, but produce markedly different image of it, which in acute and pathological cases of hemispheric dysfunction can be deficient. The reality as we grasp it is produced by a form of back-and-forth process between the two hemispheres, which may easily go out of balance if one is more dominant than the other.

With relatively little effort, it can be seen that the left-hemispheric perspective can be mapped to our P1 mode—time-less, probabilistically driven, and self-contained—whereas the right-hemispheric perspective corresponds to our P3 mode—it is present in real-time, reacts to incoming inputs as they come along and is, ideally, less bound to rules (say, to probabilistic averages). The amalgamation of P1 and P3 creates reality as we know it: a P2 reality, which once it takes place is deterministic, albeit noisy, because of the incompleteness and imperfection involved in the information gathering through P1 and P3 (see §9.7).

While it may appear as overly associative and far-fetched, the fundamentally dissimilar starting points and reasoning processes in which the present and McGilchrist's theories have reached similar conceptualizations of reality may serve to cross-validate both. Interestingly, in one of his concluding remarks McGilchrist (2009, p. 460) suggested a deeper connection, in a similar spirit to this work: "*I believe our brains not only dictate the shape of our experience we have of the world, but are likely themselves to reflect, in their structure and functioning, the nature of the universe in which they have come about.*"

We should qualify the above by noting that with regards to the example of word disambiguation (§6.4), most literature indicates that the semantic processing involved in it takes place in *Broca's area* of the left hemisphere (Rodd, 2018)—a general area for much of language processing (Turker et al., 2023). Processing ambiguity in the right hemisphere (in an analog area of Broca's area) has not been explored nearly as much, although it is known to have some involvement in that process. In recent tests of orthographic ambiguity, some sequential back and forth between the hemispheres was documented (Mizrachi et al., 2024; see also, Drijvers et al., 2025), but its relevance to spoken ambiguity and its specific processing may be too complex to readily map to any of the modes as were suggest in §6.4.

9.11 Mode transition

Theorem 1 prescribes that the modes cannot be simultaneously true, but allows them to switch by temporally following one another, as long as we entertain the possibility of observer and observed systems, in which the observer can modify the boundaries of the observed. In this way we could map the three modes to diverse situations that culminated in the claim that there should generally be a discontinuity between modes with an unknown effect. We explored two examples of this within quantum mechanics, which may be considered to be highly speculative. Even if taken as essentially correct, the dichotomous and instantaneous nature of the transition may be rejected as grossly unsatisfying, because it does not explain what the actual process is that physically happens whenever modes switch. If the boundaries are only a creation of our consciousness, then why should we ever measure any physical manifestation of it? Then, surely, the physics and its Reality should be the same in all cases, and the framing of the theorem only endows them with an uninformative and unnecessary veil of mystery.

I admit that I do not have a good counterargument for this potential criticism and would be happy to see it being picked up by someone with a fresh perspective on this matter. All I can do is submit the following, more philosophical take on the matter.

In some situations, it may seem that all three modes are equally correct, and it is only a matter of momentary convenience and ability to use whatever knowledge we have, or choice of perspective that should result in the preference of one over the other. Yet in other situations, it appears that we are truly constrained in our ability to acquire information about the system, which renders all the modes idealized. It is not impossible that we are "held captive" by the concepts of time, frequency, and determinism, which inherently and necessarily embody this contradictory Reality. Replacing these three with some other concepts may allow us to escape their grasp and obtain an image of Reality that is smoother. Escaping determinism, for example, has been particularly notorious in the sciences, as it has never been easy to generate true randomness (rather, pseudo-randomness) that is fully indeterministic (Yu et al., 2019), but it has also been difficult to produce and define systems that are neither *deterministic* nor *indeterministic*—Popper's "*cloudy clouds*" (Popper, 1965 / 1994).

Some of this discussion seems to go parallel with the problem of defining true complexity, which occupies the large space between *deterministic* and stochastic systems—both of which are relatively simple to model, in comparison with complex systems (dubbed "*problems of simplicity*" and "*problems of disorganized complexity*", respectively, by Weaver, 1948). We did not explore the mode transitions that may be found on the edge of determinism, just when *deterministic* systems turn chaotic and produce what would otherwise appear random. This dynamics is a gateway to various effects in complex systems that are characteristically unintuitive. With the exception of briefly mentioning synchronization, we did not touch this vast field here, which is left for future work.

Still, the question remains whether these three concepts are only fundamental to us as living humans with a particular neural wiring that makes our body, senses, perception, and cognition, or they describe something deeper about Reality, us, and our interrelation that cuts through our corporeal limitations.

10 Conclusion

This work has attempted to connect a few dots that trace the narrative of frequency as a concept, a physical quantity, a percept, and most speculatively, a dimension of Reality. It resulted in a theorem that ties together time, frequency, and determinism in an unintuitive manner that includes three mutually exclusive modes of Reality, one of which has frequency as a dimension that is on equal footing with time. As a dimension, it was demonstrated to serve an epistemological role, which may extend to an ontology as well, given the inherently nonlocal definition of frequency. The two other alternative modes encapsulated in the theorem—that time is not a fundamental dimension or that the universe is wholly deterministic—are no less mind bending, whether they are taken as final statements, or as allowing for switching between the modes, as was explored here using a broad gamut of examples.

It is not unlikely that the preceding argumentation will leave incredulous the few committed readers, who have made it thus far in the reading, be they from the physics, engineering, philosophy, applied mathematics, signal processing, neuroscience, or any other community. Nevertheless, it is my hope that the logical reasoning in the above has been robust enough to provoke a discussion and further exploration of the possible implications of at least a subset of the ideas and issues raised.

Afterword and acknowledgments

At the time of bringing this manuscript to completion, I no longer know whether it is a work of science. There is so much a person can know. At some point, one decides to enroll to a university. An institution that distills the knowledge that generations of humans have amassed and expressed in words, in numbers, in images, in laws. In ways of thinking. Very different ways of thinking to how industries are run, where solutions must be found according to which instruments are designed to meet target specifications. One has to find a way to mentally harmonize all these ideas in mind along with other modes of thinking from the humanities and from the very experience of being alive.

Except for a handful of fields, about all the other domains that I claim to know something about, you—the experts among you from all the various disciplines I touched and did not touch upon—you will be able to tell that I am only a visitor. I may speak your language, but I have an accent. Sometimes you may think that you could tell where I am from if you listen closely enough, but usually—hopefully—I will fool you. Or I might think that I fool you and you think that I think that I fooled you, but we both know that I fool nobody.

Which means that you will find mistakes. Only minor mistakes, I hope. If there are glaring errors, I would like to know. And in any case, I can only encourage you to find more plausible answers to the paradoxes and lacunae I highlighted, which have so far escaped scrutiny, that may end up yielding a more palatable portrait of reality according to your best judgment.

As a rule, I prefer to acknowledge all those that had shown real interest, kindness, support, friendship, generosity with ideas, and openness in discussions regarding my work—often in ways they did not themselves realize. But as with so many other things, I do not know where to put the boundary between those who should be acknowledged and those who should not. People go in and out of one's life, sometimes leaving no tangible trace, occasionally leaving behind a gaping hole in one's mind and heart. We are the sum total of all these encounters—those we met in flesh and blood and those who have left something in their image and name and words that we can converse with, unwittingly. Those we dare name and those we would rather leave unnamed. If I omitted your name or included it despite what you believe is right, please accept my sincere apology. The same goes for all those whose names I did not cite, or whose works I did not know existed—not always able to find a source—but may have come up with something reminiscent of the ideas that are articulated in this work. It has been a puzzle work, more than anything else, and I made an effort to assort as many missing pieces as I could. But, there is so much I could do. And finally, apologies to those I did cite, but feel that I grossly misunderstood and misinterpreted their work—those that would have preferred to not be cited, especially not in this very context that in itself will surely be interpreted and misinterpreted well beyond my original intent.

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A Determinism

A.1 The Fourier integral as a proxy for determinism

Throughout the text, the notion of determinism routinely creeps into the discussion about frequency. As this work reveals, determinism is essential for the understanding of the relationship between frequency, time, and Reality. Therefore, we aim to have an operational definition of determinism that can be used in all the different contexts touched upon here. We shall thus refer to the canonical definition of determinism that was given by Pierre Simon Laplace: "*Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes*"[94,95] (Laplace, 1814a, p. 4). While Laplace's definition has been criticized on several grounds (Earman, 1986), its operationalizable logic consistently coincides with the signal-analytic practice and theory that are at the heart of this work.

The main motivation for invoking determinism in this work is the near universal application of the Fourier transform (and other similar transforms) in the analysis of physical systems, waves, and

[94]The original French text is: "*Une intelligence qui pour un instant donné, connaitrait toutes les forces dont la nature est animée, et la situation respective des êtres qui la composent, si d'ailleurs elle ètait assez vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvemens des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle, et l'avenir comme le passé, serait présent à ses yeux.*" (Laplace, 1814b, p. II).

[95]Laplace crystallized earlier ideas of Leibniz about causality and prophetic intelligence (Weinert, 2016, p. 65–72).

signals—both in theory and in experiment. As long as the Fourier integral mathematically exists, perfect knowledge at one point in space of the Fourier spectrum—a time-independent function of frequency—is equivalent to perfect knowledge of its reciprocal time function over the entire time domain. Applying Laplace's definition, perfect knowledge of the time signal satisfies the requirement for determinism, as it contains all the information about the past and the future of the function, which effectively describes some *local* physics. Thus, without necessarily having to account for the various causes that generate a given arbitrary spectrum, the Fourier integral endows us with a powerful method to directly evaluate whether we can refer to a particular physical outcome as deterministic, i.e., that it appears with probability equal to one (see Footnote 12). Even if perfect knowledge of any spectrum only exists strictly theoretically (say, of a closed classical mechanical system), it is nevertheless sensible to elucidate what conditions are necessary to have this knowledge exist, if only hypothetically. Contradiction of these conditions may therefore lead us to conclude that the system is indeterministic. This is explored in the analysis mainly in § 3.4.3, § 3.5.1, and throughout the remainder of the main text.

To my best knowledge, the time-independent spectrum and, specifically, the availability of the Fourier integral have not been previously highlighted as bearers of determinism within the extensive literature on the topic in philosophy of science. Invoking this idea in the context of classical determinism seems to be warranted for the following four reasons:

1. The Fourier integral (and its inverse; Eqs. 40 and 41) can be thought of as a mathematical identity, which means that nothing in time should be able to escape its purview—but only as long as the time-independent spectrum can be determined with certainty.
2. It is a tool that can be directly used to solve a large class of (linear) partial differential equations, which may in turn have classically been the primary analytical tool that had planted the idea in the first place that determinism through physics is possible (van Strien, 2021).
3. Any physical measurement that is carried over time can be associated with a time series or a continuous signal, which, ideally, can be subjected to Fourier analysis (perhaps through more general signal processing) and is agnostic to linearity and other properties that may or may not characterize the physical system that generated it.
4. In practice, the Fourier transform appears ubiquitously in the analysis of both *deterministic* and *indeterministic* physical systems and processes.

In summary, we maintain that **perfect knowledge of the Fourier spectrum—whether attainable in theory or in practice—is tantamount to making a claim of determinism**.

A.2 The uncomfortable union between the Fourier series and the Fourier transform

Fourier analysis is primarily applied in two large classes of problems that are solved using either the Fourier series or the Fourier transform. The series is used in physical problems that are bounded and, hence, periodic in the conservative sense—without a decaying amplitude, so that the oscillations are sustained indefinitely (see comment in §3.2). The series, as the name implies, results in coefficients of a discrete series of frequencies, but is not necessarily suggestive of a claim on mapping the entire time domain as the Fourier transform does (§ 3.4.1 and § 3.4.3). The same systems can also be analyzed using the Fourier transform, wherein their boundedness (either in space or in time) would have to be represented by a matching finite-support rectangular window (§3.5.6), whose own effect enters the continuous spectrum through convolution. However, not only does it result in a more cumbersome solution, but it may also not be essential for the understanding of the physics of the problem. Hence, in the limit of very long oscillation duration relative to the period, the transient effects associated with the finiteness of the window can be safely neglected, so the Fourier series provides a better grasp of the physics of the system (Fig. 10).

Some systems exhibit physical behavior that combines what appears as both discrete and continuous spectra, so the analysis reverts to Fourier transform—the more general of the two analyses. Frequencies are then represented using the *Dirac delta function* that is more accurately considered a *generalized function* (such as a distribution or a measure), whose entire support is concentrated in an infinitesimally narrow bandwidth, reduced to the size of a point (Dirac, 1967, pp. 58–61). The delta function is defined as

$$\delta(\omega) = \begin{cases} \infty, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases} \tag{83}$$

and through its integral

$$\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1 \tag{84}$$

The most useful property of the delta function is in integration, when it appears in a product with another function $g(\omega)$

$$\int_{-\infty}^{\infty} g(\omega)\delta(\omega) d\omega = g(0) \tag{85}$$

Therefore, the inverse Fourier transform of a discrete spectrum $X(\omega) = \sum_n c_n \delta(\omega - \omega_n)$ that consists of delta functions is

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t} \sum_n c_n \delta(\omega - \omega_n) d\omega = \sum_n c_n e^{i\omega_n t} \tag{86}$$

which is the same pure sine series as in the (harmonic) Fourier series (Eq. 36). With this, the series and the transform may be united, both mathematically and conceptually—something that is commonly done in many fields. However, as is usually emphasized in introductions to the topic, the condition for absolute integrability—and thus for the Fourier transform to exist—is not satisfied with the delta function (e.g., Wiener, 1930; Middleton, 1996 / 1960; Goodman, 2017). To make it mathematically sound, the Fourier integral is replaced with a more general integral (the Riemann–Stieltjes, Lebesque, or Lebesque–Stieltjes integrals) that can absorb the discontinuity in the delta function, which normally addresses the problem.

It is worth emphasizing the unrealistic nature of the delta function, which mirrors the trouble with applying the standard (Riemann) integration, using a quote from the seminal work by Blackman and Tukey (1958, p. 256): "*Functions of time, such as* $\cos \omega_0 t$, *which represent an infinitely long past and future history of activity, are not a bit more realistic in a physical sense than are 'infinitely sharp' lines in the frequency spectrum. Similarly, functions of frequency, such as* $\exp(-i\omega t_0)$, *whose absolute values do not vanish as* $f \to \infty$, *are not a bit more realistic than impulsive 'functions' of time.*"

Mathematical rigor aside, the inclusion of discrete frequency points within the Fourier spectrum is suspect when we scrutinize its existence as a proxy for determinism, as was argued in §A.1. In §3, it was also additionally argued that perpetually oscillating systems are an idealization in classical physics for two reasons. First, because classical systems contain at least an infinitesimal damping component that will eventually cause the amplitude to decay. More critically, they are idealized because an oscillation has to have had a beginning, which provided it with the energy for oscillating in the first place (Fig. 22). Therefore, strictly speaking, analyzing these systems without considering a starting point ignores their true cause—some external force—which leads to the elimination of the transient components and the associated spectral artifacts that would be associated with its finite duration. Nevertheless, both discrete and combined discrete–continuous *deterministic* systems have been traditionally considered in physics to be fully deterministic (see Footnote 12). And indeed, given initial conditions (amplitude and phase of the oscillation), we can accurately predict the motion of such systems (e.g., a simple harmonic oscillator) at any point in time. However, the more idealization that has gone into the modeling of this oscillation (i.e., the more damping there is in the oscillation that was neglected, or the shorter is its total duration compared to the fundamental period), the larger the error of the prediction will be in the remote past and future when compared to Reality. Therefore, we reach a contradiction: we have systems that are deemed deterministic by virtue of their *deterministic* mathematical idealization, but are physically unrealizable. This makes this particular form of determinism—the one associated with pure oscillations—untenable. Driving this point further, given that true discrete oscillations never die out and have neither beginning nor end (§ 3.4.1), they extend to the infinite past and future, which implies that the extent of time—perhaps, corresponding to the age of the universe—is infinite. Reversing the order of inference, we can finally ask: Can a universe with an infinite extent of time ever be considered truly deterministic?

This issue may come across as definitional nitpicking, but we encounter it in full force in quantum mechanics, where the majority of systems are quantized in one way or another, and therefore admit discrete frequencies[96]. This is one hallmark feature that distinguishes the quantum from the classical,

[96] Moreover, the general solution of the time-independent Schrödinger equation is a series of stationary eigenstates whose energy levels (and hence frequencies in corresponding emission or absorption of photons upon level transition) have zero variance around the mean (Griffiths and Schroeter, 2018, pp. 27–28). Given the validity of the Schrödinger equation, the only way for the variance to be non-zero is if the potential is at least minimally time-dependent, so that the separation of variables that is necessary to obtain the harmonic time dependence can only be taken as an approximation.

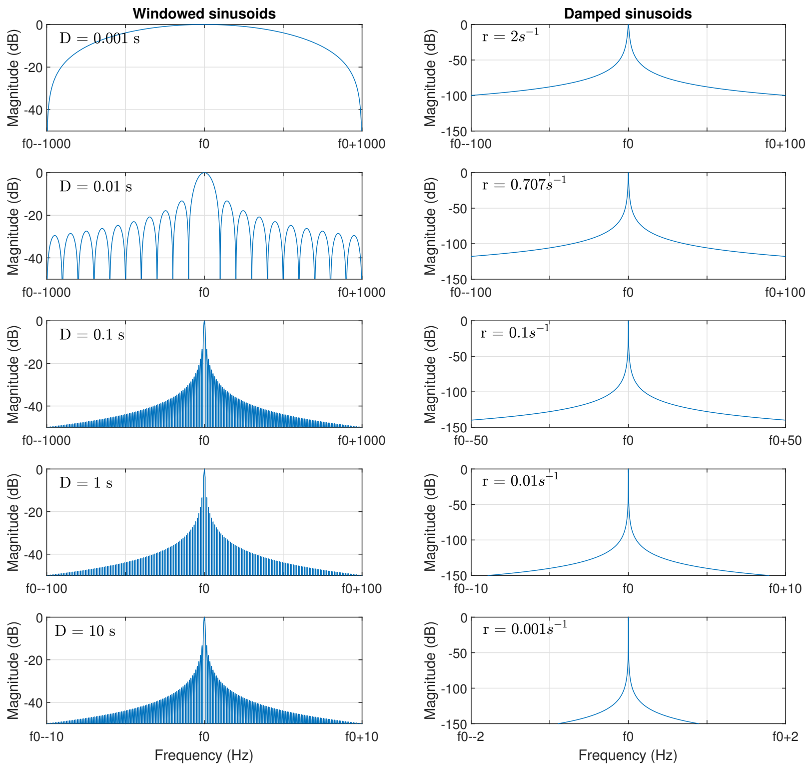

Figure 22: Two sources of infinitesimal broadening due to the finiteness of the signal duration and damping. **Left**: The Fourier transform of a cosine at frequency f_0 Hz with finite duration D (rectangular window) $x(t) = \text{rect}(t/D)\cos(\omega_0 t)$ is displayed for five different values (shortest on top; the symmetrical negative spectrum is not displayed). In line with the uncertainty principle (Eq. 45), the longer the duration of the signal or wave is, the narrower it becomes and the closer the corresponding sinc function gets to the Dirac delta function—the Fourier transform of a cosine in the limit of infinite duration: $X(\omega) \sim \text{sinc}\,[(\omega - \omega_0)D] + \text{sinc}\,[(\omega + \omega_0)D]$ (omitting the constant $1/2\pi D$ factor). **Right**: The effect on the spectrum of a minute damping constant r for signals beginning at $t = 0$ of the form $x(t) = e^{-rt}\cos(\omega_0 t)u(t)$. The corresponding spectrum of this signal is $X(\omega) \sim [r + i(\omega - \omega_0)]^{-1} + [r + i(\omega + \omega_0)]^{-1}$. More heavily damped signals are on the top and lightly damped signals are on the bottom. In both cases, the broadening effect for long / lightly-damped signals is negligible for most applications imaginable and may also be downright unmeasurable in practice. Nevertheless, it is neither mathematically nor physically zero, which is invoked here to argue for the idea that frequency is continuous and should not be taken as discrete in deterministic systems. Note that the broadening depends on the absolute values of the duration / damping and is independent of f_0. The absolute value of the Fourier transform is displayed in logarithmic, additive units, so each 10 dB drop in the magnitude is equivalent to 10 times the drop in power, emphasizing the difference in power between the peak and its flanks.

where it is understood that if a classical system appears discretized, it is only a coarse-grained idealization. Thus, we reach two additional contradictions with the reasoning above. First, we have quantum systems with fixed energy levels that have corresponding constant frequencies in the parametric sense, even though their lifetime is finite. This would violate the rules of basic signal processing and, specifically, the windowing property that should universally apply to all truncated Fourier transforms, and broaden all frequencies, even infinitesimally (§3.5.6 and Figs. 14 and 22, right). Second, such pure-state solutions are routinely considered to be deterministic, as is the Schrödinger equation itself (see §7.2). Whenever similar equations appear in classical mechanics, we know that they are an idealization that results in de-facto (ever so slight) indeterminism—even the most precise macroscopic oscillator will exhibit some transience. But if indeed in quantum systems the discrete spectra faithfully represent the physical Reality of being a truly isolated system and is not merely an idealization, then we may have at least one physical system type in existence that is both deterministic and has an infinite time extent, barring any particle decay, or another process that renders the solutions to be of finite duration. In the framework of Theorem 1, this situation corresponds to proposition P7 (deterministic Reality with parametric frequency and parametric time), which was ruled out due to contradiction that was shown in §5 (~~P7~~), and was also ruled out for classical systems in particular in the previous paragraph. However, we may resolve this paradoxical condition, if we instead use our knowledge that the quantum state is, in fact, indeterministic for other reasons (e.g., superposition eigenstate, the impossibility to measure the absolute phase of the pure state). Such a solution would constrain the system to P1, while retaining the negation of P7.

Whether pure quantum states in Reality truly admit delta-function-like infinitesimally-narrowband spectra remains an open question in this work. However, due to the measurement problem (§7) and the impossibility to directly measure the amplitude and phase of the quantum wave function, coming up with a method to precisely observe the state without perturbing the closed quantum system and its spectrum may be impossible (§7.5), leaving the question purely theoretical. That said, we may be inspired by the related controversy of the existence of quantum jumps between discrete energy levels, whose instantaneous nature bothered Schrödinger (1952a,b). In a recent experiment, *quantum trajectory theory* was employed to reconstruct the Hilbert space time dynamics of quantum jumps at a resolution of 100 ns, where it was found that they are indeed not instantaneous, although the jumps appear instantaneous if the time resolution of the measurement is made coarser (Minev et al., 2019). While the jumps were triggered spontaneously, their dynamics followed a deterministic path between the initial and final states. Although these measurements did not include or imply a corresponding time–frequency trajectory that can conclusively show evidence for transience in otherwise perfectly constant energy states, it seems reasonable that such a measurement would indeed reveal transient components.

Another case study that challenges the negation of P7 is the atomic (and the related optical) clock—a meticulously engineered quantum electronic instrument, whose precision almost defies imagination[97]. The universal reference for the atomic clock and for standardized time measurements is a specific hyperfine quantum transition of the unpaired electron in the outer shell of the caesium atoms that leads to the release of a photon, corresponding to the energy difference between the hyperfine levels ($F = 4$, $M = 0$ to $F = 3$, $M = 0$) of the ground state $^2S_{1/2}$ and is *defined* to be $|E_2 - E_1|/h = 9,192,631,770$ Hz (BIPM, 2024). The entire design of the atomic clock is geared to isolate (i.e., to physically filter) photons emitted at that exact frequency, minimizing or correcting for various biasing and spectral broadening effects such as the Doppler, Zeeman, and blackbody radiation effects (Audoin and Guinot, 2001, pp. 109–235). The clock is built so that it has a resonance at 9.192... GHz at which the corresponding transition probability of a beam of caesium atoms is maximized. A crystal oscillator excites the atoms to the particular state defined above using microwave radiation. A special detector is tuned to detect ions at that state, whose output is converted to electric current that is proportional to the number of particles detected. Any deviation from resonance causes a drop in this number, and thus a drop in probability of particles detected, which generates an error signal—an error that is proportional to a deviation in frequency—that is used to correct the electronic oscillator by virtue of a feedback loop with gain, filter, and a mixer (similar to the PLL circuit mentioned in §3.5.9). Therefore, this highly intricate instrument produces a dimensional, deterministic time output (P2) from a parametric, probabilistic quantum effect (P1), within which the transition frequency is exact. In order to get a stable output, though, the clock must harness a real-time P3 tracking mechanism, where frequency is time-dependent, and hence dimensional. Therefore, here too, the quantum effect in use is indeterministic—to us, the observers, it is still expressed as a probability and is therefore P1 and not P7.

For the purpose of this work, we assume that P7 does not exist in Reality, so that Theorem 1 remains valid with only three mutually exclusive modes of Reality and not four.

B Metaphysics

The following is an addendum to the main text, where the notions of God and Consciousness are treated head-on, using the framework of Theorem 1 and the analysis of determinism in §A.

B.1 Laplace's "demon", time, and precision

Having elucidated the relationship between determinism and time in §A.2, we can revisit Laplace's definition of determinism. We reached the conclusion that determinism with respect to particular

[97]At the time of writing, a new optical clock boasts a fantastic uncertainty of $2.5 \cdot 10^{-18}$ (Hausser et al., 2025)—the equivalent of about one second over the current estimate of the age of the universe of $13.8 \cdot 10^9$ years. This is ten orders of a magnitudes better than the very first atomic clock that had a reported uncertainty of 100 μs per day (Essen and Parry, 1955).

systems, waves, or signals—anything that can be subjected to Fourier analysis—requires finite time to be considered truly deterministic. Although we argued that the combination of infinite time and deterministic systems does not exist, we left the question open of whether such systems can truly exist in the quantum domain, as standard quantum theory implicitly predicts. In the following, let us indeed entertain the notion that for all systems in the universe, determinism requires finite time.

Laplace invoked an "*intelligence sufficiently vast*" that can analyze an arbitrarily large amount of data and obtain a perfect (i.e., zero-uncertainty) prediction and retrodiction on every physical scale of the universe. In the scientific folklore, the vast intelligence was renamed "*Laplace's demon*" (the term appeared in print at the latest in Margenau, 1931)—the first demon in a respectable tradition in the scientific gendankenexperiment literature (Weinert, 2016). Though, depending on the size of the system that is being analyzed for which a prediction is produced, referring to the vast intelligence as a demon is unnecessarily conjuring negative connotations, in addition to misestimating the required intelligence for the task. On a small-scale system, perhaps a very clever human scientist, well-equipped with the best of theories and measurement facilities, can do just as well. On a larger scale, perhaps it is future artificial intelligence (AI) that would be able to process the amount of data according to all known scientific laws that humanity has fed it with, and more. On the scale of the entire universe, the vast intelligence, by its very definition, can only be thought of as God. A demon or any other supernatural entity, would perhaps be better placed somewhere in between these categories. However, instead of insisting on its most appropriate name, let us rather assume that the entity possesses intelligence that is so much greater than the intelligence of one individual that it can be treated as infinite. We shall notate it with $intelligence_\infty$.

Similarly, the perfectly certain knowledge of past and future that the intelligence commands—the very essence of determinism—can be reformulated as perfect precision with respect to any physical quantity in the universe (cf., Popper, 1965 / 1994; Del Santo and Gisin, 2019). Let us, once again, treat it as an infinite quantity, and notate it with $precision_\infty$.

Therefore, we seem to arrive at a relation that underlies determinism comprising infinite intelligence, infinite precision, and finite time. Contrasting it with Theorem 1, this abstract relation can be immediately mapped to P2, which is the only mode of Reality that is both deterministic and has dimensional time. By symmetry, we can change the finiteness / infiniteness of the three abstract quantities—time, precision, and intelligence, and try to map them to the other two modes of Reality of the theorem. So, if we retain the infinite intelligence, assume infinite time, while making the precision finite, it naturally fits to P1, which is characterized by a probabilistic approach that necessarily gnaws on precision, in contradistinction to P2. Infinite time (or rather, infinite periodicity in P1), though, should cause the system to become indeterministic, as was argued in § A.2. The last combination would then have finite intelligence, infinite time, and infinite precision and can only be mapped to P3. The idea of finite intelligence is strange, but since this is a localized mode, centered around the present time window, which typically describes synchronized physics that can be made arbitrarily precise, there is some sense in it: whatever intelligence there is, it applies only to a narrow present window and cannot handle the entire past and future, so it cannot (or need not) be infinite—certainty not at the degree that is required by an infinite time domain.

Thus, we can summarize these three combinations of two infinite and one finite "quantities" as generators of the three modes of Reality we identified. We shall blatantly abuse the mathematical notation of the dot product operation to convey the point, remembering that (in proper mathematics) a finite quantity times an infinite quantity is infinite. Using the product form, we can reformulate the three modes of Theorem 1 with the following mutually exclusive prepositions:

P1. $precision \cdot intelligence_\infty = time_\infty (\equiv periodicity_\infty)$

P2. $time \cdot precision_\infty = intelligence_\infty$

P3. $intelligence \cdot time_\infty = precision_\infty$

where the ∞ subscript indicates an infinite quantity, property, resource, dimension, etc., whereas the absence of subscript indicates a finitude. Intuitively, these strange "equations" can be understood as constraining the number of infinities to two out of three in every given mode. The reasoning for this may be that there must be something finite in the realistic experience, which is nevertheless produced by infinities. Therefore, a corollary of this reformulation of Theorem 1 is that Reality cannot be produced with all simultaneous infinite time, precision, and intelligence.

We note that this reformulation of the theorem may clue us into the idea that an infinitely precise spectrum in P2 is equivalent to infinite intelligence: it contains all the information needed about the closed universe.

B.2 Time and intelligence

Amplifying Laplace's rather vague cues in his description of a know-all non-demonic demon, we arrived in §B.1 at a peculiar pseudo-mathematical formulation of Theorem 1, which contains various instantiations of intelligence, time, and precision. In this subsection we further this exploration by making the substitution of infinite intelligence with God. For this, we invoke one of the many attributes of God as being characterized by *omniscience*—presumably a precondition for determinism to be even hypothetically plausible. In this, we obtain three different relations between God and time, which are defined by the third—that is, by precision.

Starting again from P2, we have an "equality" between God and the "product" of time and infinite precision. Therefore, we can reduce it to the assertion that in P2 God is, literally, the *Product* of time[98]. This relation is inverted in P1, where God produces infinite time, using finite precision. We can therefore translate it to God as the *Creator* of time, or rather, periodicity. Finally, in P3 we only have a finite intelligence of God—perhaps not a full manifestation of His omniscience, although different interpretations to this statement likely exist. Now infinite precision is the product of this finite intelligence and infinite time—a switch of roles from P1. Thus, we may translate it to God being the *Subject* of time in P3. In summary,

P1. *God the Creator of Time.*

P2. *God the Product of Time.*

P3. *God the Subject of Time.*

These three markedly different relationships between God and Time prescribe fundamentally different experiences of reality. It may be that the very definition of God is the impossible union between the three, which are otherwise mutually exclusive. This definition is both paradoxical and, arguably, self-contained, which underscores the impossibility for us to conceive the meaning of God: here God is both the cause and the effect of Time, which is both the cause and the effect of God. This paradoxical statement is reminiscent of Spinoza's *causa sui* God that is both the cause of everything and a cause of itself. According to his Proposition 16, Corollary 1: "*Hence it follows that God is the efficient cause of all things which can fall under the infinite intellect*" (Spinoza, 1677 / 2001), where we can have time as one such thing—mapped well to P1. Then, according to Proposition 16, Corollary 2: "*It follows, secondly, that God is cause through Himself, and not through that which is contingent (per accidens).*" This may be mapped to P2 (or to the totality of P1, P2, and P3). Preposition 17, Corollary 3 finally states: "*Hence it follows, firstly, that there is no cause, either external to God or Within Him, which can excite Him to act except the perfection of His own nature.*" Perhaps a stretch, but it may be mapped to P3, thus mirroring the self-reflexive logic of the above formulation.

B.3 Consciousness

Now, instead of focusing on the vast intelligence, we can invert the perspective of P2, by appealing to the measurement problem of quantum mechanics (§7). One of the most controversial and least palatable interpretations of this problem entails that the wave function collapse is caused by the very act of observation by a conscious being (von Neumann, 1932 / 2018; London and Bauer, 1939 / 1983; Wigner, 1961 / 1983). Penrose (2019), in another provocative hypothesis, inverted this logic and proposed that consciousness is caused by the collapse, instead of the other way round. We argued that the measurement problem may be an inevitability, in line with Bassi and Ghirardi (2000). But we also argued that consciousness is required to make sense and define boundaries (§ 9.5). The solution of physical problems is in large part about defining the boundaries of the system that is being modeled, and thereby defining its relation with an "outside" world. While it seems that many boundaries exist in Reality and are easily defined, when it comes to the fine details of considering something to be part of the system, it usually comes down to a decision. Sometimes, the decision is straightforward. For example, a polished crystal lying on the ground is easily separable from the soil and the atmosphere surrounding it. But, a porous object (like a sponge) combines mixed media of air trapped within solid, which defies a simple boundary drawing. When inspecting living systems, the decision is even more complex. Is the volume of air that is enclosed in our nostrils part of our body? Or the air between our hairs? Are the gut microbes we carry part of us? The same

[98] We ambivalently use the English word "product" in two senses here: the result of mathematical "multiplication" and the outcome of production through work and thought.

goes even for ethereal objects such as words. Written words are easy to distinguish because of the space between them (in some languages), but in running speech, their acoustic boundaries—should they at all exist—blend in a complex neurophysiological process responsible for speech production (Daniloff and Hammarberg, 1973; Kühnert and Nolan, 1997). Examples are as abundant as there are definable things, regardless of domain. The things themselves—certainly the inanimate objects among them—do not need to draw any boundaries, but rather, boundaries emerge when we observe them, maybe through cognitive processes, heuristics, neural signal processing schemes, or other perfectly justifiable evolutionary rationales.

We argue, therefore, that consciousness is mandatory for making boundaries—there is no sense in boundaries if there is nobody there to distinguish between what is being included or excluded. And so, we fuse the ideas of von Neumann, London, Bauer, and Wigner and have it that consciousness is indeed essential for the quantum measurement, because consciousness is required for the act of observation in general. As was analyzed in §7.3, the post-measurement state of the quantum system is in P2, where it is moved to the classical domain. We thus reformulate P2 with: Consciousness as *observation*. This fits a deterministic universe worldview, in which the conscious agents that we are appear to have no control over our life, so we can, at best, observe our own life unfold.

It was argued throughout this work that Reality, as experienced by perceiving animals such as ourselves, is tri-modal. Therefore, we would like to elucidate the role that consciousness, as the boundary shifting process we possess, has in both P1 and P3. First, P1 describes isolated systems that are dominated by pre-existent probability. Any role that we can have here is strictly passive, as we cannot open these systems or interact with them directly. Not even observe them. This corresponds to a mode of *being*.

Lastly, in P3 there is a sense of agency, because it specifically relates to open systems, which can interact, synchronize, be controlled, feed-back, and feed-forward to themselves, and perhaps be less influenced by the shadow of the omniscient intelligence, which is only finite here. Therefore, the most appropriate mode of consciousness here may be *participation*.

We therefore summarize the three modes of reality according to the roles that Consciousness may be having in each:

P1. *Being.*

P2. *Observation.*

P3. *Participation.*

Once again, the three modes are mutually exclusive, although it would not be incorrect to say that reality can be all three together and that they do not contradict one another. Nevertheless, this formulation relates to us as conscious agents much more intuitively than previous formulations of the theorem, so the mutual exclusivity at any given moment is perhaps easier to accept in this case. Moreover, the distinction between the three modes creates an opening for free will to enter the experience of reality via P3—something which has been notoriously difficult to conciliate with the scientific and philosophical views that subscribe either to random indeterminism (P1) or complete determinism (P2), even when it verges on chaos that only appears indeterministic (Sapolsky, 2023).

B.4 God, consciousness, and free will

P3 was associated above with finite intelligence as well as with participation, this mode may provide a resolution for the "*foreknowledge problem*" of an omniscient God that does not seem to allow for free will to exist (attributed to Boethius; Pike, 1965; Hunt and Zagzebski, 2022). Omniscience only shows in P1 and P2, but not in P3, where participation is key. Inasmuch as participation calls—and Reality enables—for even a limited amount of free will, then it is not in violation of God's omniscience. This appears to be a corollary of the mutual exclusivity of the three modes of Reality elucidated in this work.

We shall follow suit in our final foray into the realm of metaphysics and juxtapose the last two formulations pertaining to God and to Consciousness, in order to obtain a clearer handle on their interrelationship as was done for P3. This final reformulation of Theorem 1 is provided without argumentation and with no further interpretation.

P1. *God the Creator of Time* ⇔ *Consciousness is.*

P2. *God the Product of Time* ⇔ *Consciousness observes.*

P3. *God the Subject of Time* ⇔ *Consciousness participates.*

References

Abramovitch, Daniel. Phase-locked loops: A control centric tutorial. In *Proceedings of the 2002 American control conference (IEEE cat. No. CH37301)*, volume 1, pages 1–15. IEEE, 2003.

Adlam, Emily. Two roads to retrocausality. *Synthese*, 200(5):422, 2022.

Aertsen, AMHJ, Johannesma, Peter IM, and Hermes, DJ. Spectro-temporal receptive fields of auditory neurons in the grassfrog. II. Analysis of the stimulus-event relation for tonal stimuli. *Biological Cybernetics*, 38(4):235–248, 1980a.

Aertsen, AMHJ, Olders, JHJ, and Johannesma, PIM. Spectro-temporal receptive fields of auditory neurons in the grassfrog. I. Characterization of tonal and natural stimuli. *Biological Cybernetics*, 38(3):223–234, 1980b.

Aharonov, Y and Bohm, D. Remarks on the possibility of quantum electrodynamics without potentials. *Physical Review*, 125(6):2192, 1962.

Aharonov, Yakir and Bohm, David. Significance of electromagnetic potentials in the quantum theory. *Physical Review*, 115(3):485, 1959.

Aharonov, Yakir, Albert, David Z, and Vaidman, Lev. How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. *Physical Review Letters*, 60 (14):1351, 1988.

Aharonov, Yakir, Cohen, Eliahu, and Rohrlich, Daniel. Comment on “role of potentials in the Aharonov-Bohm effect”. *Physical Review A*, 92(2):026101, 2015.

Aharonov, Yakir, Cohen, Eliahu, and Rohrlich, Daniel. Nonlocality of the Aharonov-Bohm effect. *Physical Review A*, 93(4):042110, 2016.

Aharonov, Yakir, Cohen, Eliahu, Colombo, Fabrizio, Landsberger, Tomer, Sabadini, Irene, Struppa, Daniele C, and Tollaksen, Jeff. Finally making sense of the double-slit experiment. *Proceedings of the National Academy of Sciences*, 114(25):6480–6485, 2017.

Aleksandrova, Antoniya, Borish, Victoria, and Wootters, William K. Real-vector-space quantum theory with a universal quantum bit. *Physical Review A*, 87(5):052106, 2013.

Allman, Melissa J, Teki, Sundeep, Griffiths, Timothy D, and Meck, Warren H. Properties of the internal clock: First-and second-order principles of subjective time. *Annual Review of Psychology*, 65:743–771, 2014.

Aoki, Takao, Takei, Nobuyuki, Yonezawa, Hidehiro, Wakui, Kentaro, Hiraoka, Takuji, Furusawa, Akira, and van Loock, Peter. Experimental creation of a fully inseparable tripartite continuous-variable state. *Physical Review Letters*, 91(8):080404, 2003.

Arkani-Hamed, Nima. Quantum mechanics and spacetime in the 21st century. https://www.youtube.com/watch?v=U47kyV4TMnE, 2014. Lecture given at the Perimeter Institute for Theoretical Physics, on Nov. 6, 2014.

Aspect, Alain. Bell's inequality test: more ideal than ever. *Nature*, 398(6724):189–190, 1999.

Aspect, Alain, Grangier, Philippe, and Roger, Gérard. Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A new violation of Bell's inequalities. *Physical Review Letters*, 49(2):91, 1982.

Attneave, Fred and Olson, Richard K. Pitch as a medium: A new approach to psychophysical scaling. *The American Journal of Psychology*, pages 147–166, 1971.

Audoin, Claude and Guinot, Bernard. *The Measurement of Time: Time, Frequency and the Atomic Clock*. Cambridge University Press, Cambridge, UK, 2001. Translated from French by Stephen Lyle.

Auffarth, Benjamin. Understanding smell—The olfactory stimulus problem. *Neuroscience & Biobehavioral Reviews*, 37(8):1667–1679, 2013.

Ballentine, Lelslie E. Einstein's interpretation of quantum mechanics. *American Journal of Physics*, 40(12):1763–1771, 1972.

Ballentine, Leslie E. The statistical interpretation of quantum mechanics. *Reviews of Modern Physics*, 42(4):358, 1970.

Barbour, Julian. *The End of Time: The Next Revolution in our Understanding of the Universe*. Phoenix, London, UK, 1999.

Barrett, Jonathan. Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality. *Physical Review A*, 65(4):042302, 2002.

Bassi, Angelo and Ghirardi, GianCarlo. A general argument against the universal validity of the superposition principle. *Physics Letters A*, 275(5-6):373–381, 2000.

Bedrosian, Edward. A product theorem for Hilbert transforms. *Proceedings of the IEEE*, 51(5): 868–869, 1963.

Bell, Alexander Graham. Improvement in transmitters and receivers for electric telegraphs. US Patent 161,739, 1875.

Bell, J. S. Beables for quantum field theory. Technical Report CERN-TH.4035/84, CERN, Geneva, October 1984.

Bell, John S. On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

Bell, John S. The theory of local beables. Technical Report TH-2053-CERN, CERN, 1975. Reprinted in Bell, J. S., "Speakable and Unspeakable in Quantum Mechanics. Collected papers on quantum philosophy", pp. 52–62, Cambridge University Press, 1987.

Bell, John S. Are there quantum jumps? Technical report, 1987. Reprinted in Bell, J. S., "Speakable and Unspeakable in Quantum Mechanics. Collected papers on quantum philosophy", pp. 201–212, Cambridge University Press, 1987.

Bendat, Julius S and Piersol, Allan G. *Random Data: Analysis and Measurement Procedures.* John Wiley & Sons, Hoboken, NJ, 4th ed. edition, 2011.

Benedetti, Giovanni Battista. *Diversarum speculationum mathematicarum, & physicarum liber.* apud Haeredem Nicolai Bevilaquae, Turin, 1585.

BIPM, Bureau international des poids et mesures. *Le Système international d'unités / The International System of Units (SI).* 9th, v3.01 edition, 2024.

Blackman, R. B. and Tukey, J. W. The measurement of power spectra from the point of view of communications engineering — Part I. *Bell System Technical Journal*, 37(1):185–282, 1958.

Blinchikoff, Herman J. and Zverev, Anatol I. *Filtering in the time and frequency domains.* Scitech Publishing Inc., Rayleigh, NC, 2001.

Block, Eric, Jang, Seogjoo, Matsunami, Hiroaki, Sekharan, Sivakumar, Dethier, Bérénice, Ertem, Mehmed Z, Gundala, Sivaji, Pan, Yi, Li, Shengju, Li, Zhen, Lodge, Stephene N., Ozbil, Mehmet, Jiang, Huihong, Penalba, Sonia F., Batista, Victor S., and Zhuang, Hanyi. Implausibility of the vibrational theory of olfaction. *Proceedings of the National Academy of Sciences*, 112(21): E2766–E2774, 2015.

Boashash, Boualem. Estimating and interpreting the instantaneous frequency of a signal. I. fundamentals. *Proceedings of the IEEE*, 80(4):520–538, 1992.

Bode, Henrdrik W. *Network Analysis and Feedback Amplifier Design.* D. Van Nostrand Company, Inc., Princeton, NJ, 1945. 12th printing, 1957.

Boersma, Paul. Accurate short-term analysis of the fundamental frequency and the harmonics-to-noise ratio of a sampled sound. In *Proceedings of the Institute of Phonetic Sciences, University of Amesterdam*, volume 17, pages 97–110, 1993.

Bohm, David. *Quantum Theory.* Dover Publications, Inc., Mineola, NY, 1951. Reprinted edition 1989.

Bohm, David. A suggested interpretation of the quantum theory in terms of "hidden" variables. I. *Physical Review*, 85(2):166–179, 1952a.

Bohm, David. A suggested interpretation of the quantum theory in terms of "hidden" variables. II. *Physical Review*, 85(2):180–193, 1952b.

Bohm, David and Aharonov, Yakir. Discussion of experimental proof for the paradox of Einstein, Rosen, and Podolsky. *Physical Review*, 108(4):1070, 1957.

Bohm, David and Hiley, Basil J. *The Undivided Universe: An Ontological Interpretation of Quantum Theory.* Routledge, Oxon, UK and New York, NY, 1993.

Bohm, David J and Hiley, Basil J. On the intuitive understanding of nonlocality as implied by quantum theory. *Foundations of Physics*, 5(1):93–109, 1975.

Bohr, Niels. Discussion with Einstein on epistemological problems in atomic physics. In Schilpp, Paul Arthur, editor, *Albert Einstein: Philosopher-Scientist*, volume 7, pages 201–241. MJF Books, New York, NY, 1970.

Bolanowski Jr, Stanley J, Gescheider, George A, Verrillo, Ronald T, and Checkosky, Christin M. Four channels mediate the mechanical aspects of touch. *The Journal of the Acoustical society of America*, 84(5):1680–1694, 1988.

Boltzmann, Ludwig. Theoretical physics and philosophical problems. selected writings. D. Reidel Publishing Company, Dordrecht, Holland, 1974 / 1899. From Populäire Schiften, Essay 16, 1899.

Translated from German by Paul Foulkes.

Born, Max. Quantenmechanik der Stoßvorgänge. *Zeitschrift für Physik*, 38(11):803–827, 1926. Translated by D. H. Delphenich.

Born, Max, Born, Hedwig, and Einstein, Albert. *The Born-Einstein Letters: Correspondence between Albert Einstein and Max and Hedwig Born from 1916 / 1955*. Macmillan, 1971. Translated by Irene Born.

Born, Max, Wolf, Emil, Bhatia, A. B., Clemmow, P. C., Gabor, D., Stokes, A. R., Taylor, A. M., Wayman, P. A., and Wilcock, W. L. *Principles of Optics*. Cambridge University Press, Cambridge, United Kingdom, 7th (expanded) edition, 2003.

Borsten, Leron, Jubb, Ian, and Kells, Graham. Impossible measurements revisited. *Physical Review D*, 104(2):025012, 2021.

Brillouin, Léon. Über die Fortpflanzung des Lichtes in dispergierenden Medien. *Annalen der Physik*, 349(10):203–240, 1914. Translated to English by Dr. E. Erlbach and reprinted in Leon Brillouin "Wave Propagation and Group Velocity", pp. 43–84, 1960, Academic Press, Inc.

Brillouin, Léon. *Wave Propagation and Group Velocity*. Academic Press, Inc., New York, NY, 1960.

Brown, R Hanbury and Twiss, Richard Q. Correlation between photons in two coherent beams of light. *Nature*, 177(4497):27–29, 1956.

Brunner, Nicolas, Cavalcanti, Daniel, Pironio, Stefano, Scarani, Valerio, and Wehner, Stephanie. Bell nonlocality. *Reviews of Modern Physics*, 86(2):419–478, 2014.

Bub, Jeffrey. *Interpreting the Quantum World*. Cambridge University Press, Cambridge, UK, 1999.

Budroni, Costantino, Cabello, Adán, Gühne, Otfried, Kleinmann, Matthias, and Larsson, Jan-Åke. Kochen-Specker contextuality. *Reviews of Modern Physics*, 94(4):045007, 2022.

Burnet, John. *Early Greek Philosophy*. A & C Black, London, 3rd edition, 1920.

Burnham, David C and Weinberg, Donald L. Observation of simultaneity in parametric production of optical photon pairs. *Physical Review Letters*, 25(2):84, 1970.

Busch, Paul. "No information without disturbance": quantum limitations of measurement. In Myrvold, Wayne C and Christian, Joy, editors, *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle. Essays in Honour of Abner Shimony*, pages 229–256. Springer Science+Business Media B.V., 2009.

Byrne, Denis, Dillon, Harvey, Tran, Khanh, Arlinger, Stig, Wilbraham, Keith, Cox, Robyn, Hagerman, Bjorn, Hetu, Raymond, Kei, Joseph, Lui, C., Kiessling, Jurgen, Nasser Notby, M., Nasser, Nasser H. A., El Kholy, Wafaa A. H., Nakanishi, Yasuko, Oyer, Herbert, Powell, Richard, Stephens, Daffyd, Meredith, Rhys, Sirimanna, Tony, Tavartkiladze, George, Frolenkov, Gregory I., Westerman, Soren, and Ludvigsen, Carl. An international comparison of long-term average speech spectra. *The Journal of the Acoustical Society of America*, 96(4):2108–2120, 1994.

Cabello, Adán. Interpretations of quantum theory: A map of madness. In Lombardi, Olimpia, Fortin, Sebastian, Holik, Federico, and López, Cristian, editors, *What is Quantum Information?*, pages 138—-144. Cambridge University Press, Cambridge, UK, 2017.

Cajori, Florian. Origins of fourth dimension concepts. *The American Mathematical Monthly*, 33(8): 397–406, 1926.

Campbell, G. A. Electric wave-filter, 1917. US Patent 1,227,113; Application filed July 15, 1915.

Campbell, George A. Physical theory of the electric wave-filter. *The Bell System Technical Journal*, 1(2):1–32, 1922.

Capecchi, Giulia and Capecchi, Danilo. Between acoustics and music: Two letters of Giovanni Battista Benedetti to Cipriano de Rore. In *Società italiana degli storici della fisica e dell'astronomia: proceedings of the SISFA 42nd Annual Conference: Perugia, 26–29 September 2022, Department of Physics and Geology, University of Perugia*, pages 231–238. Pisa University Press, 2023.

Carleson, Lennart. On convergence and growth of partial sums of Fourier series. *Acta Mathematica*, 116:135–157, 1966.

Carson, John R. Notes on the theory of modulation. *Proceedings of the Institute of Radio Engineers*, 10(1):57–64, 1922.

Carson, John R and Fry, Thornton C. Variable frequency electric circuit theory with application to the theory of frequency-modulation. *Bell System Technical Journal*, 16(4):513–540, 1937.

Catani, Lorenzo, Leifer, Matthew, Schmid, David, and Spekkens, Robert W. Why interference phenomena do not capture the essence of quantum theory. *Quantum*, 7:1119, 2023.

Caves, Carlton M, Fuchs, Christopher A, and Schack, Rüdiger. Unknown quantum states: The quantum de Finetti representation. *Journal of Mathematical Physics*, 43(9):4537–4559, 2002.

Chambers, RG. Shift of an electron interference pattern by enclosed magnetic flux. *Physical Review Letters*, 5(1):3, 1960.

Chen, Yi-Chia, Chang, Andrew, Rosenberg, Monica D., Feng, Derek, Scholl, Brian J., and Trainor, Laurel J. "Taste typicality" is a foundational and multi-modal dimension of ordinary aesthetic experience. *Current Biology*, 32:1–6, 2022.

Chiao, Raymond Y, Kwia, Paul G, and Steinberg, Aephraim M. Quantum non-locality in two-photon experiments at Berkeley. *Quantum and Semiclassical Optics: Journal of the European Optical Society Part B*, 7(3):259, 1995.

Clauser, John F, Horne, Michael A, Shimony, Abner, and Holt, Richard A. Proposed experiment to test local hidden-variable theories. *Physical Review Letters*, 23(15):880, 1969.

Cocciaro, Bruno, Faetti, Sandro, and Fronzoni, Leone. Improved lower bound on superluminal quantum communication. *Physical Review A*, 97(5):052124, 2018.

Coffman, Valerie, Kundu, Joydip, and Wootters, William K. Distributed entanglement. *Physical Review A*, 61(5):052306, 2000.

Cohen, Leon. Time-frequency distributions—A review. *Proceedings of the IEEE*, 77(7):941–981, 1989.

Cohen, Leon. *Time-Frequency Analysis*. Prentice Hall PTR, Upper Saddle River, NJ, 1995.

Cohen-Tannoudji, Claude, Diu, Bernard, and Laloë, Frank. *Quantum Mechanics*. WILEY-VCH Verlag GmbH & Co., 2nd edition, 2020. Translated from French by Susan Reid Hemley, Nicole Ostrowsky, and Dan Ostrowsky.

Cole, KC. Time, space obsolete in new view of universe. *Los Angeles Times*, Nov 16, 1999.

Conrad, Carol. Context effects in sentence comprehension: A study of the subjective lexicon. *Memory & Cognition*, 2:130–138, 1974.

Cooley, James W. The re-discovery of the fast Fourier transform algorithm. *Microchimica Acta*, 93: 33–45, 1987.

Cooley, James W. and Tukey, John W. An algorithm for the machine calculation of complex Fourier series. *Mathematics of Computation*, 19(90):297–301, 1965.

Couch II, Leon W. *Digital and Analog Communication Systems*. Pearson Education Inc., Upper Saddle River, NJ, 8th edition, 2013.

Couteau, Christophe. Spontaneous parametric down-conversion. *Contemporary Physics*, 59(3):291–304, 2018.

Cowen, Ron. Space, time, entanglement. *Nature*, 527(7578):290–293, 2015.

Cramer, John G. The transactional interpretation of quantum mechanics. *Reviews of Modern Physics*, 58(3):647, 1986.

Dalton, Bryan J. Can quantum theory be underpinned by a non-local hidden variable theory? *Physica Scripta*, 99(7):075018, 2024.

Daniloff, Raymond G and Hammarberg, Robert E. On defining coarticulation. *Journal of Phonetics*, 1(3):239–248, 1973.

Davies, P. C. W. and Brown, J. R. *The Ghost in the Atom: A Discussion of the Mysteries of Quantum Physics*. Cambridge University Press, New York, 1986.

de Broglie, Louis. *An Introduction to the Study of Wave Mechanics*. Methuen & Co. Ltd., 1930. Translated from French by H. T. Flint.

Debnath, Lokenath. *Wavelet Transforms and Their Applications*. Birkhäuser, Boston, MA, 2002.

Del Santo, Flavio and Gisin, Nicolas. Physics without determinism: Alternative interpretations of classical physics. *Physical Review A*, 100(6):062107, 2019.

Descartes, René. *Discourse on the Method and Related Writings*. Penguin Books, London, England, 1637 / 2004. Translated from French by Desmond M. Clarke.

DeWitt, Bryce S. Quantum theory without electromagnetic potentials. *Physical Review*, 125(6): 2189, 1962.

Dirac, P. A. M. *The Principles of Quantum Mechanics*. Oxford University Press, London, United Kingdom, 4th edition, 1967.

Dostrovsky, Sigalia. Early vibration theory: Physics and music in the seventeenth century. *Archive for History of Exact Sciences*, pages 169–218, 1975.

Doyle, Bob. Nonlocality, 2015. Retrieved from https://www.informationphilosopher.com/

problems/nonlocality/ on 1 January 2025.

Drijvers, Linda, Small, Steven L, and Skipper, Jeremy I. Language is widely distributed throughout the brain. *Nature Reviews Neuroscience*, 26(189):1, 2025.

Drullman, Rob, Festen, Joost M, and Plomp, Reinier. Effect of temporal envelope smearing on speech reception. *The Journal of the Acoustical Society of America*, 95(2):1053–1064, 1994.

Drummond, Brian. Understanding quantum mechanics: A review and synthesis in precise language. *Open Physics*, 17(1):390–437, 2019.

Duffieux, P. M. *The Fourier Transform and Its Applications to Optics.* John Wiley & Sons, 2nd edition, 1946 / 1983. Translated from French, "L'intégrale de Fourier et ses applications á l'optique", Masson, Editeur, Paris, 1970. First published in 1946.

Dürr, S, Nonn, T, and Rempe, G. Origin of quantum-mechanical complementarity probed by a 'which-way' experiment in an atom interferometer. *Nature*, 395(6697):33–37, 1998.

Earman, John. *A Primer on Determinism.* D. Reidel Publishing Company, Dordrecht, Holland, 1986.

Eberhard, P. H. A realistic model for quantum theory with a locality property. In Schommers, W., editor, *Quantum Theory and Pictures of Reality: Foundations, Interpretations, and New Aspects*, pages 169–215. Springer-Verlag, Berlin Heidelberg, 1989.

Ehrenberg, Werner and Siday, Raymond E. The refractive index in electron optics and the principles of dynamics. *Proceedings of the Physical Society. Section B*, 62(1):8, 1949.

Einstein, A. Method for the determinination of the statistical values of observations concerning quantities subject to irregular fluctuations. *Archives de Sciences Physiques et Naturelles*, 37: 254–256, 1914. Reproduced in the IEEE ASSP magazine, Vol. 4, p. 6, October 1987.

Einstein, Albert. Relativity and the problem of space. In *Ideas and Opinions*, pages 360–377. Bonanza Books, New York, NY, 1954.

Einstein, Albert and Rosen, Nathan. The particle problem in the general theory of relativity. *Physical Review*, 48(1):73, 1935.

Einstein, Albert, Podolsky, Boris, and Rosen, Nathan. Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777, 1935.

Ellis, George FR, Meissner, Krzysztof A, and Nicolai, Hermann. The physics of infinity. *Nature Physics*, 14(8):770–772, 2018.

Essen, Louis and Parry, Jack VL. An atomic standard of frequency and time interval: A caesium resonator. *Nature*, 176(4476):280–282, 1955.

Euler, Leonhard. De novo genere oscillationum (On a new class of oscillations). *Commentarii academiae scientiarum Petropolitanae*, 11, 1750 / 2021. Originally written in 1739. Translated from Latin by Sylvio R Bistafa, arXiv preprint arXiv:2105.03319, 2021.

Everett III, Hugh. "Relative state" formulation of quantum mechanics. *Reviews of Modern Physics*, 29(3):454, 1957.

Ewald, Paul P. Die Berechnung optischer und elektrostatischer Gitterpotentiale. *Annalen der Physik*, 369(3):253–287, 1921.

Fiore, Tiziana and Pellerito, Claudia. Infrared absorption spectroscopy. In Agnello, Simonpietro, editor, *Spectroscopy for Materials Characterization*, pages 129–167. John Wiley & Sons, Inc., Hoboken, NJ, 2021.

Fishman, Ronald S. Gordon Holmes, the cortical retina, and the wounds of war. *Documenta Ophthalmologica*, 93(1):9–28, 1997.

Flanders, Martha. Functional somatotopy in sensorimotor cortex. *Neuroreport*, 16(4):313–316, 2005.

Flandrin, Patrick. *Explorations in Time-Frequency Analysis.* Cambridge University Press, Cambridge, United Kingdom, 2018.

Fletcher, Harvey. Auditory patterns. *Reviews of Modern Physics*, 12(1):47–65, 1940.

Fodor, Jerry A. *The Modularity of Mind. An Essay on Faculty Psychology.* Bradford Books, MIT Press, Cambridge, MA and London, England, 1983.

Folland, Gerald B. *Fourier Analysis and Its Applications.* Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California, 1992.

Folland, Gerald B and Sitaram, Alladi. The uncertainty principle: A mathematical survey. *Journal of Fourier Analysis and Applications*, 3:207–238, 1997.

Fourier, Jean Baptiste Joseph. *The Analytical Theory of Heat.* Cambridge University Press, New York, 1822 / 2009. Translated from the French, "Théorie analytique de la chaleur", by Alexander

Freeman and originally appeared in English in 1878.

Franson, JD. Nonlocal cancellation of dispersion. *Physical Review A*, 45(5):3126, 1992.

French, Norman R and Steinberg, John C. Factors governing the intelligibility of speech sounds. *The Journal of the Acoustical Society of America*, 19(1):90–119, 1947.

Fuchs, Christopher A. QBism, the perimeter of quantum Bayesianism. *arXiv preprint arXiv:1003.5209*, 2010.

Fuchs, Christopher A and Peres, Asher. Quantum theory needs no 'interpretation'. *Physics Today*, 53(3):70–71, 2000.

Fuchs, Christopher A, Mermin, N David, and Schack, Rüdiger. An introduction to QBism with an application to the locality of quantum mechanics. *American Journal of Physics*, 82(8):749–754, 2014.

Gabor, Dennis. Theory of communication. Part 1: The analysis of information. *Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering*, 93(26):429–457, 1946.

Galapon, Eric. Pauli's theorem and quantum canonical pairs: The consistency of a bounded, self–adjoint time operator canonically conjugate to a Hamiltonian with non–empty point spectrum. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 458(2018):451–472, 2002.

Galileo, Galilei. *Dialogues Concerning Two New Sciences.* Dover Publishing, New York, NY, 1638 / 1914. Translated by Henry Crew & Alfonso de Salvio.

Gardner, W. Introduction to Einstein's contribution to time-series analysis. *IEEE ASSP Magazine*, 4(4):4–5, 1987.

Garuccio, Augusto and Selleri, Franco. Nonlocal interactions and Bell's inequality. *Nuovo Cim., B*, 36(2):176–185, 1976.

Gaspard, Pierre. *Chaos, Scattering and Statistical Mechanics.* Cambridge University Press, Cambridge, United Kingdom, 1998.

Genovese, Marco. Interpretations of quantum mechanics and measurement problem. *Advanced Science Letters*, 3(3):249–258, 2010.

Genovese, Marco and Gramegna, Marco. Quantum correlations and quantum non-locality: A review and a few new ideas. *Applied Sciences*, 9(24):5406, 2019.

Ghazal, Ammar, Yuan, Yi, Wang, Cheng-Xiang, Zhang, Yan, Yao, Qi, Zhou, Hongrui, and Duan, Weiming. A non-stationary IMT-advanced MIMO channel model for high-mobility wireless communication systems. *IEEE Transactions on Wireless Communications*, 16(4):2057–2068, 2017.

Ghirardi, Gian-Carlo, Rimini, Alberto, and Weber, Tullio. A general argument against superluminal transmission through the quantum mechanical measurement process. *Lettere al Nuovo Cimento (1971-1985)*, 27(10):293–298, 1980.

Ghirardi, Gian Carlo, Rimini, Alberto, and Weber, Tullio. Unified dynamics for microscopic and macroscopic systems. *Physical Review D*, 34(2):470, 1986.

Ghirardi, Gian Carlo, Grassi, R, Rimini, A, and Weber, T. Experiments of the EPR type involving CP-violation do not allow faster-than-light communication between distant observers. *Europhysics Letters*, 6(2):95, 1988.

Ghosh, R and Mandel, Leonard. Observation of nonclassical effects in the interference of two photons. *Physical Review Letters*, 59(17):1903, 1987.

Gibbs, J Willard. Fourier's series. *Nature*, 59(1522):200, 1898.

Gibbs, J Willard. Fourier's series. *Nature*, 59(1539):606, 1899.

Gisin, Nicolas. Quantum nonlocality: How does nature do it? *Science*, 326(5958):1357–1358, 2009.

Giustina, Marissa, Versteegh, Marijn AM, Wengerowsky, Sören, Handsteiner, Johannes, Hochrainer, Armin, Phelan, Kevin, Steinlechner, Fabian, Kofler, Johannes, Larsson, Jan-Åke, Abellán, Carlos, Amaya, Waldimar, Pruneri, Valerio, Mitchell, Morgan W., Beyer, Jörn, Gerrits, Thomas, Lita, Adriana E., Shalm, Lynden K., Nam, Sae Woo, Scheidl, Thomas, Ursin, Rupert, Wittmann, Bernhard, and Zeilinger, Anton. Significant-loophole-free test of Bell's theorem with entangled photons. *Physical Review Letters*, 115(25):250401, 2015.

Glauber, Roy J. The quantum theory of optical coherence. *Physical Review*, 130(6):2529, 1963.

Gleick, James. *Time Travel: A History.* 4th Estate, London, UK, 2016.

Goggin, Michael E, Almeida, Marcelo P, Barbieri, Marco, Lanyon, Benjamin P, O'Brien, Jeremy L, White, Andrew G, and Pryde, Geoff J. Violation of the Leggett–Garg inequality with weak measurements of photons. *Proceedings of the National Academy of Sciences*, 108(4):1256–1261, 2011.

Goldstein, Herbert, Poole Jr., Charles P., and Safko, John L. *Classical Mechanics*. Pearson Education Limited, Harlow, UK, 3rd edition, 2014.

Gonzalez-Ayala, Julian, Cordero, Rubén, and Angulo-Brown, F. Is the (3+ 1)-d nature of the universe a thermodynamic necessity? *Europhysics Letters*, 113(4):40006, 2016.

Goodman, Joseph W. *Introduction to Fourier Optics*. W. H. Freeman and Company, New York, NY, 4th edition, 2017.

Green, Michael B., Schwarz, John H., and Witten, Edward. *Superstring theory: Volume 1, Introduction*. Cambridge University Press, Cambridge, UK, 1987.

Greenberger, Daniel M, Horne, Michael A, Shimony, Abner, and Zeilinger, Anton. Bell's theorem without inequalities. *American Journal of Physics*, 58(12):1131–1143, 1990.

Griffiths, David J. and Schroeter, Darrell F. *Introduction to Quantum Mechanics*. Cambridge University Press, Cambridge, UK, 3rd edition, 2018.

Griffiths, Robert B. Consistent histories and the interpretation of quantum mechanics. *Journal of Statistical Physics*, 36:219–272, 1984.

Griffiths, Robert B. *Consistent Quantum Theory*. Cambridge University Press, 2003.

Griffiths, Robert B. Nonlocality claims are inconsistent with Hilbert-space quantum mechanics. *Physical Review A*, 101(2):022117, 2020.

Gröblacher, Simon, Paterek, Tomasz, Kaltenbaek, Rainer, Brukner, Časlav, Żukowski, Marek, Aspelmeyer, Markus, and Zeilinger, Anton. An experimental test of non-local realism. *Nature*, 446 (7138):871–875, 2007.

Grondin, Simon. Timing and time perception: A review of recent behavioral and neuroscience findings and theoretical directions. *Attention, Perception, & Psychophysics*, 72(3):561–582, 2010.

Hall, Michael JW and Branciard, Cyril. Measurement-dependence cost for Bell nonlocality: Causal versus retrocausal models. *Physical Review A*, 102(5):052228, 2020.

Hamilton, William R. *Proceedings of the Royal Irish Academy*, 1:267–270 and 341–349, 1839.

Han, Chong, Wang, Yiqin, Li, Yuanbo, Chen, Yi, Abbasi, Naveed A, Kürner, Thomas, and Molisch, Andreas F. Terahertz wireless channels: A holistic survey on measurement, modeling, and analysis. *IEEE Communications Surveys & Tutorials*, 24(3):1670–1707, 2022.

Hance, Jonte R, Rarity, John, and Ladyman, James. Weak values and the past of a quantum particle. *Physical Review Research*, 5(2):023048, 2023.

Harris, SE. Nonlocal modulation of entangled photons. *Physical Review A*, 78(2):021807, 2008.

Hausser, HN, Keller, J, Nordmann, T, Bhatt, NM, Kiethe, J, Liu, H, Richter, IM, von Boehn, M, Rahm, J, Weyers, S, Benkler, E., Lipphardt, B., Dörscher, S, Stahl, K., Klose, J., Lisdat, C., Filzinger, M., Huntemann, N., Peik, E., and Mehlstäubler, T. E. $^{115}In^{+}$-$^{172}Yb^{+}$ Coulomb crystal clock with 2.5×10^{-18} systematic uncertainty. *Physical Review Letters*, 134(2):023201, 2025.

Healy, John J, Kutay, M Alper, Ozaktas, Haldun M, and Sheridan, John T. *Linear Canonical Transforms: Theory and Applications*, volume 198. Springer, 2016.

Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 43:172–198, 1927.

Heisenberg, Werner. Quantum-theoretical re-interpretation of kinematic and mechanical relations. *Zeitschrift für Physik*, 33:879–893, 1925.

Hensen, Bas, Bernien, Hannes, Dréau, Anaïs E, Reiserer, Andreas, Kalb, Norbert, Blok, Machiel S, Ruitenberg, Just, Vermeulen, Raymond FL, Schouten, Raymond N, Abellán, Carlos, Amaya, W., Pruneri, V., Mitchell, M. W., Markham, M., Twitchen, D. J., Elkouss, D., Wehner, S., Taminiau, T. H., and Hanson, R. Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature*, 526(7575):682–686, 2015.

Heraclitus. *Fragments. A Text and Translation with a Commentary by T. M. Robinson*. University of Toronto Press, Toronto, Buffalo, London, 1991.

Hewitt, Edwin and Hewitt, Robert E. The Gibbs-Wilbraham phenomenon: An episode in Fourier analysis. *Archive for History of Exact Sciences*, pages 129–160, 1979.

Hoffman, Donald D. Spacetime is doomed: Time is an artifact. *Timing & Time Perception*, 12(2): 189–191, 2024.

Hoffman, Donald D, Singh, Manish, and Prakash, Chetan. The interface theory of perception. *Psychonomic Bulletin & Review*, 22(6):1480–1506, 2015.

Hogaboam, Thomas W and Perfetti, Charles A. Lexical ambiguity and sentence comprehension. *Journal of Verbal Learning and Verbal Behavior*, 14(3):265–274, 1975.

Hossenfelder, Sabine and Palmer, Tim. Rethinking superdeterminism. *Frontiers in Physics*, 8:139, 2020.

Howard, Don. Who invented the "Copenhagen Interpretation"? A study in mythology. *Philosophy of Science*, 71(5):669–682, 2004.

Huang, Norden E, Shen, Zheng, Long, Steven R, Wu, Manli C, Shih, Hsing H, Zheng, Quanan, Yen, Nai-Chyuan, Tung, Chi Chao, and Liu, Henry H. The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis. *Proceedings of the Royal Society of London A*, 454(1971):903–995, 1998.

Huang, Norden E, Shen, Zheng, and Long, Steven R. A new view of nonlinear water waves: The Hilbert spectrum. *Annual Review of Fluid Mechanics*, 31(1):417–457, 1999.

Huey, Erica L, Turecek, Josef, Delisle, Michelle M, Mazor, Ofer, Romero, Gabriel E, Dua, Malvika, Sarafis, Zoe K, Hobble, Alexis, Booth, Kevin T, Goodrich, Lisa V, Corey, David P., and Ginty, David D. The auditory midbrain mediates tactile vibration sensing. *Cell*, 188(1):104–120, 2025.

Hume, David. Of the component parts of our reasonings concerning cause and effect. In *A Treatise of Human Nature*. Wordsworth Editions Limited, Ware, England, 1740. From the volume: The Essential Philosophical Writings (2011).

Hume, David. Skeptical doubts concerning the operations of the understanding. In *An Enquiry Concerning Human Understanding*. Wordsworth Editions Limited, Ware, England, 1748. From the volume: The Essential Philosophical Writings (2011).

Hunt, David and Zagzebski, Linda. Foreknowledge and Free Will. In Zalta, Edward N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Summer 2022 edition, 2022.

Huygens, Cristiaan. Sea clocks (sympathy of clocks). Part V. In Pikovsky, Arkady, Rosenblum, Michael, and Kurths, Jürgen, editors, *Synchronization: A Universal Concept in Nonlinear Sciences*, pages 358–361. Cambridge University Press, Cambridge, United Kingdom, 1665 / 2001. Translated from Latin by Dorothea Prell.

Iannilli, Emilia, Noennig, Nina, Hummel, Thomas, and Schoenfeld, Ariel M. Spatio-temporal correlates of taste processing in the human primary gustatory cortex. *Neuroscience*, 273:92–99, 2014.

Jackson, John David. *Classical Electrodynamics*. John Wiley & Sons, Inc., Hoboken, NJ, 3rd edition, 1999.

Jammer, Max. *The Philosophy of Quantum Mechanics: The Interpretations of Quantum Mechanics in Historical Perspective*. John Wiley & Sons, Inc., 1974.

Jansson, Lina. Action at a distance: From Newton's gravity to quantum theory. In *Journal of Physics: Conference Series. HAPP Centre: 10th Anniversary Commemorative Volume*, volume 2877, page 012075. IOP Publishing, 2024.

Jenkins, Gwilym M. and Watts, Donald G. *Spectral Analysis and its Applications*. Holden-Day, Inc., San Francisco, CA, 1968.

Johansson, R. S., Landström, U., and Lundström, R. Responses of mechanoreceptive afferent units in the glabrous skin of the human hand to sinusoidal skin displacements. *Brain Research*, 244(1): 17–25, 1982.

Johnson, J. B. Thermal agitation of electricity in conductors. *Physical Review*, 32(1):97–109, 1928.

Kaluza, Theodor. Zum unitätsproblem der physik. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)*, pages 966–972, 1921. Revised translation by V. T. Toth, based in part on translation by T. Muta, HUPD-8401, March 1984, Department of Physics University of Hiroshima.

Kang, Kicheon. Locality of the Aharonov-Bohm-Casher effect. *Physical Review A*, 91(5):052116, 2015.

Kang, Kicheon. Gauge invariance of the local phase in the Aharonov-Bohm interference: Quantum electrodynamic approach. *Europhysics Letters*, 140(4):46001, 2022.

Kant, Immanuel. *Prolegomena to any future metaphysics that will be able to come forward as science.* Hackett Publishing Company, Inc., Indianapolis, IN, 1783 / 2001. Translated from German by Paul Carus; extensively revised by James W. Ellington.

Kasturi, Kalyan, Loizou, Philipos C, Dorman, Michael, and Spahr, Tony. The intelligibility of speech with "holes" in the spectrum. *The Journal of the Acoustical Society of America*, 112(3):1102–1111, 2002.

Keller, Andreas and Vosshall, Leslie B. A psychophysical test of the vibration theory of olfaction. *Nature Neuroscience*, 7(4):337–338, 2004.

Keller, Timothy E, Rubin, Morton H, Shih, Yanhua, and Wu, Ling-An. Theory of the three-photon entangled state. *Physical Review A*, 57(3):2076, 1998.

Kendall, Maurice G. *The Advanced Theory of Statistics: Volume I.* Charles Griffin & Company Limited, London, United Kingdom, 1945.

Kennard, Earle H. Zur Quantenmechanik einfacher Bewegungstypen. *Zeitschrift für Physik*, 44(4): 326–352, 1927.

Khintchine, Alexander. Korrelationstheorie der stationären stochastischen Prozesse. *Mathematische Annalen*, 109(1):604–615, 1934.

Khrennikov, Andrei. Get rid of nonlocality from quantum physics. *Entropy*, 21(8):806, 2019.

Klauder, John R, Price, AC, Darlington, Sidney, and Albersheim, Walter J. The theory and design of chirp radars. *Bell System Technical Journal*, 39(4):745–808, 1960.

Klein, Oskar. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik*, 37 (12):895–906, 1926.

Kloeden, Peter E and Rasmussen, Martin. *Nonautonomous Dynamical Systems.* American Mathematical Society, Providence, Rhode Island, 2011.

Klug, Achim and Grothe, Benedikt. Ethological stimuli. In Rees, Adrian and Palmer, Alan R, editors, *The Oxford Handbook of Auditory Science: The Auditory Brain*, volume 2, pages 173–192. Oxford University Press, New York, USA, 2010.

Kühnert, Barbara and Nolan, Francis. The origin of coarticulation. *Forschungsberichte des Instituts für Phonetik und Sprachliche Kommunikation der Universität München*, 35:61–75, 1997.

Kupczynski, Marian. Is the moon there if nobody looks: Bell inequalities and physical reality. *Frontiers in Physics*, 8:273, 2020.

Küpfmüller, Karl. Über Einschwingvorgange in Wellenfiltern (transient phenomena in wave filters). *Elektrische Nachrichten-Technik*, 1:141–152, 1924.

Kwiat, Paul G, Mattle, Klaus, Weinfurter, Harald, Zeilinger, Anton, Sergienko, Alexander V, and Shih, Yanhua. New high-intensity source of polarization-entangled photon pairs. *Physical Review Letters*, 75(24):4337, 1995.

Lacey, Michael T. Carleson's theorem: Proof, complements, variations. *Publicacions Matematiques*, pages 251–307, 2004.

Lackner, James R and DiZio, Paul. Vestibular, proprioceptive, and haptic contributions to spatial orientation. *Annual Reviews in Psychology*, 56:115–147, 2005.

Land, Edwin H. The retinex theory of color vision. *Scientific American*, 237(6):108–129, 1977.

Land, Edwin H and McCann, John J. Lightness and retinex theory. *Journal of the Optical Society of America*, 61(1):1–11, 1971.

Land, Kate and Magueijo, João. Examination of evidence for a preferred axis in the cosmic radiation anisotropy. *Physical Review Letters*, 95(7):071301, 2005.

Landau, L. D. and Lifshitz, E. M. *Statistical Physics. Part 1*, volume 5. Pergamon Press Ltd., 3rd, revised and enlarged edition, 1980. Translated from Russian by J. B. Sykes and M. J.Kearsley from the 3rd edition, 1976.

Landauer, Rolf. The physical nature of information. *Physics Letters A*, 217(4-5):188–193, 1996.

Langner, Gerald D. *The Neural Code of Pitch and Harmony.* Cambridge University Press, Cambridge, United Kingdom, 2015.

Laplace, M. Le Compte. *A Philosophical Essay on Probabilities.* Dover Publications, Inc., New York, 1814a. Translation from the sixth French edition by Frederick Wilson Truscott and Frederick Lincoln Emory (1951).

Laplace, Pierre-Simon. *Théorie analytique des probabilités.* Mme Ve Courcier, Paris, France, 2nd edition, 1814b.

Lee, Brian K., Mayhew, Emily J., Sanchez-Lengeling, Benjamin, Wei, Jennifer N., Qian, Wesley W., Little, Kelsie A., Andres, Matthew, Nguyen, Britney B., Moloy, Theresa, Yasonik, Jacob, Parker, Jane K., Gerkin, Richard C., Mainland, Joel D., and Wiltschko, Alexander B. A principal odor map unifies diverse tasks in olfactory perception. *Science*, 381(6661):999–1006, 2023.

Leggett, Anthony J. Nonlocal hidden-variable theories and quantum mechanics: An incompatibility theorem. *Foundations of Physics*, 33:1469–1493, 2003.

Li, Benliang, Hewak, Daniel W, and Wang, Qi Jie. The transition from quantum field theory to one-particle quantum mechanics and a proposed interpretation of Aharonov–Bohm effect. *Foundations of Physics*, 48:837–852, 2018.

Lighthill, Michael James and Whitham, Gerald Beresford. On kinematic waves II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London A*, 229(1178):317–345, 1955.

London, Fritz and Bauer, Edmond. The theory of observation in quantum mechanics. In Wheeler, John Archibald and Zurek, Wojciech Hubert, editors, *Quantum Theory and Measurement*, pages 217–259. Princeton, New Jersey, 1939 / 1983. Originally published as "La théorie de l'observation en mécanique quantique", in No. 775 of Actualités scientifiques et industrielles: Exposés de physique générale, publiés sous la direction de Paul Langevin, Hermann, Paris (1939). Translated to English by A. Shimony, J. A. Wheeler, W. H. Zurek, and J. McGrath and S. McLean McGrath.

Lüke, Hans Dieter. The origins of the sampling theorem. *IEEE Communications Magazine*, 37(4): 106–108, 1999.

Lundeen, Jeff S, Sutherland, Brandon, Patel, Aabid, Stewart, Corey, and Bamber, Charles. Direct measurement of the quantum wavefunction. *Nature*, 474(7350):188–191, 2011.

Maccone, Lorenzo and Sacha, Krzysztof. Quantum measurements of time. *Physical Review Letters*, 124(11):110402, 2020.

MacLean, Jean-Philippe W, Donohue, John M, and Resch, Kevin J. Direct characterization of ultrafast energy-time entangled photon pairs. *Physical Review Letters*, 120(5):053601, 2018.

Malcolm Dyson, G. The scientific basis of odour. *Chemistry and Industry*, 57(28):647–651, 1938.

Maldacena, Juan. The large-N limit of superconformal field theories and supergravity. *International Journal of Theoretical Physics*, 38(4):1113–1133, 1999.

Maldacena, Juan and Susskind, Leonard. Cool horizons for entangled black holes. *Fortschritte der Physik*, 61(9):781–811, 2013.

Mandel, L. Interpretation of instantaneous frequencies. *American Journal of Physics*, 42(10):840–846, 1974.

Mandel, Leonard and Wolf, Emil. *Optical Coherence and Quantum Optics*. Cambridge University Press, Cambridge, United Kingdom, 1995.

Marburger, John H. A historical derivation of Heisenberg's uncertainty relation is flawed. *American Journal of Physics*, 76(6):585–587, 2008.

Margenau, Henry. Causality and modern physics. *The Monist*, pages 1–36, 1931.

Markopoulou, Fotini and Smolin, Lee. Quantum theory from quantum gravity. *Physical Review D*, 70(12):124029, 2004.

Marletto, Chiara and Vedral, Vlatko. Aharonov-Bohm phase is locally generated like all other quantum phases. *Physical Review Letters*, 125(4):040401, 2020.

McAdams, Stephen. The perceptual representation of timbre. In Siedenburg, Kai, Saitis, Charalampos, McAdams, Stephen, Fay, Arthur N, and Popper, Richard R., editors, *The Human Auditory Cortex*, volume 69, pages 23–57. Springer Nature Switzerland AG, Cham, Switzerland, 2019.

McGilchrist, Iain. *The Master and His Emissary: The Divided Brain and the Making of the Western World*. Yale University Press, New Haven and London, 2009.

McKenna, Terence. An ecology of souls. https://ia600502.us.archive.org/22/items/377TMcKennaSoulEcology/377-TMcKennaSoulEcology.mp3, 1989. The quote appears circa 13:00 on a tape-recorded version of a workshop given by McKenna in June, 1989, on a Wednesday, somewhere in the USA. It was published as Episode #377 of the Psychedelic Salon podcast series by Larry "Lorenzo" Hagerty at https://psychedelicsalon.com/podcast-377-an-ecology-of-souls/ (accessed 16.3.2025).

McTaggart, J. M. E. The unreality of time. In Le Poidevin, Robin and MacBeath, Murray, editors, *The Philosophy of Time*. Oxford University Press, Oxford, UK, 1927 / 1993. Reprint taken from Chapter 33 in McTaggart's book, "The Nature of Existence. II", Cambridge University Press (1927), but similar arguments had already appeared in 1908.

Mead, C Alden. Possible connection between gravitation and fundamental length. *Physical Review*, 135(3B):B849, 1964.

Melia, Fulvio. The seemingly preferred cosmic frame. *Physica Scripta*, 97(4):045001, 2022.

Mermin, N David. Commentary: Quantum mechanics: Fixing the shifty split. *Physics Today*, 65

(7):8–10, 2012.

Mersenne, Marin. *Harmonie universelle, contenant la théorie et la pratique de la musique*. Sebastien Cramoisy, Paris, France, 1636.

Méthot, André Allan and Scarani, Valerio. An anomaly of non-locality. *arXiv preprint quant-ph/0601210*, 2006.

Middleton, David. Doppler effects for randomly moving scatterers and platforms. *The Journal of the Acoustical Society of America*, 61(5):1231–1250, 1977.

Middleton, David. *An Introduction to Statistical Communication Theory*. IEEE Press, Piscataway, NJ, 1996 / 1960. 2nd reprint edition; originally published in 1960.

Minev, Zlatko K, Mundhada, Shantanu O, Shankar, Shyam, Reinhold, Philip, Gutiérrez-Jáuregui, Ricardo, Schoelkopf, Robert J, Mirrahimi, Mazyar, Carmichael, Howard J, and Devoret, Michel H. To catch and reverse a quantum jump mid-flight. *Nature*, 570(7760):200–204, 2019.

Minkowski, H. Raum und zeit. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 14:75–88, 1908. Translated from German by W. Perrett and G. B. Jeffery and available in the collection "The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity by H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl", published by Dover Inc., New York, NY, 1952.

Mizrachi, Nofar, Eviatar, Zohar, Peleg, Orna, and Bitan, Tali. Inter-and intra-hemispheric interactions in reading ambiguous words. *Cortex*, 171:257–271, 2024.

Mollon, J. D. The origins of modern color science. In Shevell, Steven K., editor, *The Science of Color*, pages 1–39. Elsevier, The Optical Society of America, Oxford, United Kingdom, 2nd edition, 2003.

Moreva, Ekaterina, Brida, Giorgio, Gramegna, Marco, Giovannetti, Vittorio, Maccone, Lorenzo, and Genovese, Marco. Time from quantum entanglement: An experimental illustration. *Physical Review A*, 89(5):052122, 2014.

Morris, Michael S, Thorne, Kip S, and Yurtsever, Ulvi. Wormholes, time machines, and the weak energy condition. *Physical Review Letters*, 61(13):1446, 1988.

Morse, Philip M and Bolt, Richard H. Sound waves in rooms. *Reviews of Modern Physics*, 16(2): 69, 1944.

Morse, Philip M. and Ingard, K. Uno. *Theoretical Acoustics*. Princeton University Press, Princeton, NJ, 1968.

Muga, J. G., Mayato, R. Sala, and Egusquiza, Inigo L. Introduction. In Muga, J. G., Mayato, R. Sala, and Egusquiza, Inigo L., editors, *Time in Quantum Mechanics*, pages 1–28. Springer-Verlag, Berlin Heidelberg 2002, 2002.

Neumann, Sebastian Philipp, Buchner, Alexander, Bulla, Lukas, Bohmann, Martin, and Ursin, Rupert. Continuous entanglement distribution over a transnational 248 km fiber link. *Nature Communications*, 13(1):6134, 2022.

Newton, Isaac. *The Principia: Mathematical Principles of Natural Philosophy*. University of California Press, Oakland, California, 1687 / 1999. Translated from Latin by I. Bernard Cohen and Anne Whitman.

Nordström, Gunnar. Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen. *Physik. Zeitschr. XV*, pages 504–506, 1914. Translated by Frank Borg, as "On the possibility of unifying the electromagnetic and gravitational fields", arXiv preprint: arXiv:physics/0702221v1.

Nyquist, Harry. Thermal agitation of electric charge in conductors. *Physical Review*, 32(1):110–113, 1928.

Obradovic, Gojko. Personal Communication, 2024.

Onifer, William and Swinney, David A. Accessing lexical ambiguities during sentence comprehension: Effects of frequency of meaning and contextual bias. *Memory & Cognition*, 9:225–236, 1981.

Oppenheim, Alan V. and Schafer, Ronald W. *Discrete-Time Signal Processing*. Pearson Higher Education, Inc., Upper Saddle River, NJ, 3rd edition, 2009.

Osborne, Tobias J and Verstraete, Frank. General monogamy inequality for bipartite qubit entanglement. *Physical review letters*, 96(22):220503, 2006.

Ozawa, Masanao. Heisenberg's original derivation of the uncertainty principle and its universally valid reformulations. *Current Science*, pages 2006–2016, 2015.

Page, Chester H. Instantaneous power spectra. *Journal of Applied Physics*, 23(1):103–106, 1952.

Palisca, Claude V. *Humanism in Italian Renaissance Musical Thought*. Yale University Press, New Haven and London, 1985.

Palisca, Claude V. *Studies in the History of Italian Music and Music Theory*. Oxford University Press, Inc., New York, 1994.

Papageorgiou, Maria and Fraser, Doreen. Eliminating the 'impossible': Recent progress on local measurement theory for quantum field theory. *Foundations of Physics*, 54(3):1–75, 2024.

Pauli, Wolfgang. *General Principles of Quantum Mechanics*. Springer-Verlag Berlin Heidelberg, 1958 / 1980. Translated by P. Achuthan and K Venkatesan.

Pearle, Philip and Rizzi, Anthony. Quantized vector potential and alternative views of the magnetic Aharonov-Bohm phase shift. *Physical Review A*, 95(5):052124, 2017.

Penfield, Wilder and Boldrey, Edwin. Somatic motor and sensory representation in the cerebral cortex of man as studied by electrical stimulation. *Brain*, 60(4):389–443, 1937.

Penrose, Roger. Sir roger penrose – why explore cosmos and consciousness? https://www.youtube.com/watch?v=dlv8DVb6e0Q, 2019. Interviewed by Robert Lawrence Kuhn as part of the series "Closer to Truth". Relevant segment begins circa 13:00.

Peres, Asher. *Quantum Theory: Concepts and Methods*. Kluwer Academic Publishers, 2002.

Peres, Asher and Rosen, Nathan. Quantum limitations on the measurement of gravitational fields. *Physical Review*, 118(1):335, 1960.

Piacentini, Fabrizio, Avella, Alessio, Rebufello, Enrico, Lussana, Rudi, Villa, Federica, Tosi, Alberto, Gramegna, Marco, Brida, Giorgio, Cohen, Eliahu, Vaidman, Lev, Degiovanni, Ivo P., and Genovese, Marco. Determining the quantum expectation value by measuring a single photon. *Nature Physics*, 13(12):1191–1194, 2017.

Picinbono, Bernard. On instantaneous amplitude and phase of signals. *IEEE Transactions on Signal Processing*, 45(3):552–560, 1997.

Pickles, James O. *An Introduction to the Physiology of Hearing*. Emerald Group Publishing Limited, Bingley, United Kingdom, 4th edition, 2012.

Pike, Nelson. Divine omniscience and voluntary action. *The Philosophical Review*, 74(1):27–46, 1965.

Pikovsky, Arkady, Rosenblum, Michael, and Kurths, Jürgen. *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge University Press, Cambridge, United Kingdom, 2001. Cambridge Nonlinear Science Series 12.

Planck, Max. Über irreversible Strahlungsvorgänge. Fünfte Mittheilung. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 440–480, 1899.

Poincaré, Henri. The measure of time. In Čapek, Milič, editor, *The Concepts of Space and Time: Their Structure and Their Development*, pages 317–327. Springer-Science+Business Media, B.V., Dordrecht, Holland, 1913 / 1976. Originally published in 1913.

Popescu, Sandu. Dynamical quantum non-locality. *Nature Physics*, 6(3):151–153, 2010.

Popescu, Sandu and Rohrlich, Daniel. Quantum nonlocality as an axiom. *Foundations of Physics*, 24(3):379–385, 1994.

Popper, Karl. Of clouds and clocks: An approach to the problem of rationality and the freedom of man. In *Objective Knowledge: An Evolutionary Approach*, pages 206–255. Oxford University Press, Oxford, United Kingdom, revised edition 1979; eighth impression edition, 1965 / 1994. Originally presented as the second Arthur Holly Compton Memorial Lecture, Washington University on 21 Apr. 1965.

Potter, Ralph Kimball. Visible patterns of sound. *Science*, 102(2654):463–470, 1945.

Pressnitzer, Daniel, Meddis, Ray, Delahaye, Roel, and Winter, Ian M. Physiological correlates of comodulation masking release in the mammalian ventral cochlear nucleus. *Journal of Neuroscience*, 21(16):6377–6386, 2001.

Priestley, M. B. *Spectral Analysis and Time Series. Volume 2. Multivariate Series*. Academic Press Inc., London, UK, 1981.

Proakis, John G. and Salehi, Masoud. *Fundamentals of Communication Systems*. Pearson Education, Inc., Upper Saddle River, NJ, 2nd edition, 2014.

Rayleigh, Lord. XCII. Remarks concerning Fourier's theorem as applied to physical problems. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 24(144):864–869, 1912.

Renou, Marc-Olivier, Trillo, David, Weilenmann, Mirjam, Le, Thinh P, Tavakoli, Armin, Gisin, Nicolas, Acín, Antonio, and Navascués, Miguel. Quantum theory based on real numbers can be experimentally falsified. *Nature*, 600(7890):625–629, 2021.

Rhodes, J Elmer. Microscope imagery as carrier communication. *Journal of the Optical Society of America*, 43(10):848–852, 1953.

Rice, Caitlin A, Beekhuizen, Barend, Dubrovsky, Vladimir, Stevenson, Suzanne, and Armstrong, Blair C. A comparison of homonym meaning frequency estimates derived from movie and television subtitles, free association, and explicit ratings. *Behavior Research Methods*, 51:1399–1425, 2019.

Ringbauer, Martin, Duffus, Ben, Branciard, Cyril, Cavalcanti, Eric G, White, Andrew G, and Fedrizzi, Alessandro. Measurements on the reality of the wavefunction. *Nature Physics*, 11(3): 249–254, 2015.

Ringbauer, Martin, Giarmatzi, Christina, Chaves, Rafael, Costa, Fabio, White, Andrew G, and Fedrizzi, Alessandro. Experimental test of nonlocal causality. *Science Advances*, 2(8):e1600162, 2016.

Robertson, Howard Percy. The uncertainty principle. *Physical Review*, 34(1):163, 1929.

Rodd, Jennifer. Lexical ambiguity. In Rueschemeyer, Shirley-Ann and Gaskell, M. Gareth, editors, *Oxford Handbook of Psycholinguistics*, pages 120–144. Oxford University Press Oxford, 2nd edition, 2018.

Rosen, Nathan. On waves and particles. *Journal of the Elisha Mitchell Scientific Society*, 61(1/2): 67–73, 1945.

Roskies, Adina L. The binding problem. *Neuron*, 24(1):7–9, 1999.

Rovelli, Carlo. Relational quantum mechanics. *International Journal of Theoretical Physics*, 35: 1637–1678, 1996.

Rovelli, Carlo. *The Order of Time*. Penguin Random House, UK, 2019.

Rowland Adams, Joe, Newman, Julian, and Stefanovska, Aneta. Distinguishing between deterministic oscillations and noise. *The European Physical Journal Special Topics*, 232(20):3435–3457, 2023.

Ruben, Robert J. The developing concept of tonotopic organization of the inner ear. *Journal of the Association for Research in Otolaryngology*, pages 1–20, 2020.

Salart, Daniel, Baas, Augustin, Branciard, Cyril, Gisin, Nicolas, and Zbinden, Hugo. Testing the speed of 'spooky action at a distance'. *Nature*, 454(7206):861–864, 2008.

Saldanha, Pablo L. Local description of the Aharonov–Bohm effect with a quantum electromagnetic field. *Foundations of Physics*, 51(1):6, 2021.

Sandoval, Steven and De Leon, Phillip L. The instantaneous spectrum: A general framework for time-frequency analysis. *IEEE Transactions on Signal Processing*, 66(21):5679–5693, 2018.

Santos, E and Gonzalo, I. Microscopic theory of the Aharonov-Bohm effect. *Europhysics Letters*, 45 (4):418, 1999.

Sapolsky, Robert M. *Determined: A Science without Free Will*. Penguin Press, New York, 2023.

Scarani, Valerio and Gisin, Nicolas. Superluminal hidden communication as the underlying mechanism for quantum correlations: Constraining models. *Brazilian Journal of Physics*, 35:328–332, 2005.

Schlosshauer, Maximilian, Kofler, Johannes, and Zeilinger, Anton. A snapshot of foundational attitudes toward quantum mechanics. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 44(3):222–230, 2013.

Schmid, David, Fraser, Thomas C, Kunjwal, Ravi, Sainz, Ana Belen, Wolfe, Elie, and Spekkens, Robert W. Understanding the interplay of entanglement and nonlocality: motivating and developing a new branch of entanglement theory. *Quantum*, 7:1194, 2023.

Schneider, Gerald E. Two visual systems. *Science*, 163(3870):895–902, 1969.

Schraffenberger, Hanna and van der Heide, Edwin. Sonically tangible objects. In *xCoAx 2015: Proceedings of the Third Conference on Computation, Communication, Aesthetics and X., Glasgow, Scotland*, pages 233–248, 2015.

Schrödinger, E. Are there quantum jumps? Part I. *The British Journal for the Philosophy of Science*, 3(10):109–123, 1952a.

Schrödinger, E. Are there quantum jumps? Part II. *The British Journal for the Philosophy of Science*, 3(11):233–242, 1952b.

Schrödinger, Erwin. An undulatory theory of the mechanics of atoms and molecules. *Physical Review*, 28(6):1049, 1926.

Schrödinger, Erwin. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften*, 23 (50):844–849, 1935. Translated to English by John D. Trimmer under "The Present Situation in Quantum Mechanics: A Translation of Schrödinger's 'Cat Paradox' Paper", Proceedings of the American Philosophical Society, Vol. 124, No. 5 (Oct. 10, 1980), pp. 323–338.

Schvaneveldt, Roger W, Meyer, David E, and Becker, Curtis A. Lexical ambiguity, semantic context, and visual word recognition. *Journal of Experimental Psychology: Human Perception and Performance*, 2(2):243, 1976.

Schwartz, Mischa, Bennett, William R., and Stein, Seymour. *Communication Systems and Techniques*. IEEE Press Inc., New York, NY, 1995.

Scully, Marian O, Englert, Berthold-Georg, and Walther, Herbert. Quantum optical tests of complementarity. *Nature*, 351(6322):111–116, 1991.

Sensarn, S, Yin, GY, and Harris, SE. Observation of nonlocal modulation with entangled photons. *Physical Review Letters*, 103(16):163601, 2009.

Seshadri, Suparna, Lu, Hsuan-Hao, Leaird, Daniel E, Weiner, Andrew M, and Lukens, Joseph M. Complete frequency-bin Bell basis synthesizer. *Physical Review Letters*, 129(23):230505, 2022.

Shalm, Lynden K, Hamel, Deny R, Yan, Zhizhong, Simon, Christoph, Resch, Kevin J, and Jennewein, Thomas. Three-photon energy–time entanglement. *Nature Physics*, 9(1):19–22, 2013.

Shalm, Lynden K, Meyer-Scott, Evan, Christensen, Bradley G, Bierhorst, Peter, Wayne, Michael A, Stevens, Martin J, Gerrits, Thomas, Glancy, Scott, Hamel, Deny R, Allman, Michael S, Coakley, Kevin J., Dyer, Shellee D., Hodge, Carson, Lita, Adriana E., Verma, Varun B., Lambrocco, Camilla, Tortorici, Edward, Migdall, Alan L., Zhang, Yanbao, Kumor, Daniel R., Farr, William H., Marsili, Francesco, Shaw, Matthew D., Stern, Jeffrey A., Abellán, Carlos, Amaya, Waldimar, Pruneri, Valerio, Jennewein, Thomas, Mitchell, Morgan W., Kwiat, Paul G., Bienfang, Joshua C., Mirin, Richard P., Knill, Emanuel, and Nam, Sae Woo. Strong loophole-free test of local realism. *Physical Review Letters*, 115(25):250402, 2015.

Shannon, Claude E. A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423, 623–656, 1948.

Shannon, Claude E. Communication in the presence of noise. *Proceedings of the IRE*, 37(1):10–21, 1949.

Sharma, Rajib, Vignolo, Leandro, Schlotthauer, Gastón, Colominas, Marcelo A, Rufiner, H Leonardo, and Prasanna, SRM. Empirical mode decomposition for adaptive AM-FM analysis of speech: A review. *Speech Communication*, 88:39–64, 2017.

Shekel, J. Instantaneous frequency. *Proceedings of the Institute of Radio Engineers*, 41(4):548–548, 1953.

Shepard, Roger N. Perceptual-cognitive universals as reflections of the world. *Psychonomic Bulletin & Review*, 1(1):2–28, 1994.

Shih, Yanhua. Entangled biphoton source-property and preparation. *Reports on Progress in Physics*, 66(6):1009, 2003.

Shin, Kihong and Hammond, Joseph. *Fundamentals of Signal Processing for Sound and Vibration Engineers*. John Wiley & Sons, West Sussex, England, 2008.

Sivasundaram, Sujeevan and Nielsen, Kristian Hvidtfelt. Surveying the attitudes of physicists concerning foundational issues of quantum mechanics. *arXiv preprint arXiv:1612.00676*, 2016.

Slepian, David. On bandwidth. *Proceedings of the IEEE*, 64(3):292–300, 1976.

Slepian, David. Some comments on Fourier analysis, uncertainty and modeling. *SIAM review*, 25 (3):379–393, 1983.

Smolin, Lee. *Einstein's Unfinished Revolution: The Search for What Lies Beyond the Quantum*. Penguin Press, New York, N.Y., 2019.

Sommerfeld, Arnold. Über die Fortpflanzung des Lichtes in dispergierenden Medien. *Annalen der Physik*, 349(10):177–202, 1914. Translated to English by Dr. E. Erlbach and reprinted in Leon Brillouin "Wave Propagation and Group Velocity", pp. 17–42, 1960, Academic Press, Inc.

Sommerfeld, Arnold. *Partial Differential Equations in Physics*. Academic Press Inc., Publishers, New York, N.Y., 1949. translated by Ernst G. Straus.

Sorkin, Rafael D. Impossible measurements on quantum fields. *arXiv preprint gr-qc/9302018*, 1993.

Spasskiĭ, BI and Moskovskiĭ, AV. Nonlocality in quantum physics. *Soviet Physics Uspekhi*, 27(4): 273, 1984.

Spinoza, Benedict. *Ethics*. Wordsworth Editions Limited, Ware, England, 1677 / 2001. Translated from Latin by W. H. White; revised by A. H. Stirling.

Stapp, Henry Pierce. The Copenhagen interpretation. *American Journal of Physics*, 40(8):1098–1116, 1972.

Steeneken, Herman and Houtgast, Tammo. The temporal envelope spectrum of speech and its significance in room acoustics. In *Proc. 11th International Congress on Acoustics Lyon-Touslouse*, volume 7, pages 85–88, 1983.

Steinbauer, Martin, Molisch, Andreas F, and Bonek, Ernst. The double-directional radio channel. *IEEE Antennas and Propagation Magazine*, 43(4):51–63, 2001.

Steinberg, Aephraim M, Kwiat, Paul G, and Chiao, Raymond Y. Dispersion cancellation and high-resolution time measurements in a fourth-order optical interferometer. *Physical Review A*, 45(9): 6659, 1992a.

Steinberg, AM, Kwiat, PG, and Chiao, RY. Dispersion cancellation in a measurement of the single-photon propagation velocity in glass. *Physical Review Letters*, 68(16):2421, 1992b.

Stueckelberg, Ernst CG. Quantum theory in real Hilbert space. *Helvetica Physica Acta*, 33(727–752): 458, 1960.

Susskind, Leonard. Copenhagen vs Everett, teleportation, and ER=EPR. *Fortschritte der Physik*, 64(6-7):551–564, 2016.

Swinney, David A. Lexical access during sentence comprehension: (Re)consideration of context effects. *Journal of Verbal Learning and Verbal Behavior*, 18(6):645–659, 1979.

Talbot, William H, Darian-Smith, Ian, Kornhuber, Hans H, and Mountcastle, Vernon B. The sense of flutter-vibration: Comparison of the human capacity with response patterns of mechanoreceptive afferents from the monkey hand. *Journal of Neurophysiology*, 31(2):301–334, 1968.

Tammaro, Elliott. Why current interpretations of quantum mechanics are deficient. *arXiv preprint arXiv:1408.2093*, 2014.

Tao, Ran and Zhao, Juan. Uncertainty principles and the linear canonical transform. In Healy, John J, Kutay, M Alper, Ozaktas, Haldun M, and Sheridan, John T, editors, *Linear Canonical Transforms: Theory and Applications*, volume 198, pages 97–111. Springer Science + Business Media, New York, 2016.

Tegmark, Max. On the dimensionality of spacetime. *Classical and Quantum Gravity*, 14(4):L69, 1997.

Thomsen, Knud. Timelessness strictly inside the quantum realm. *Entropy*, 23(6):772, 2021.

Titchmarsh, E. C. *Introduction to the Theory of Fourier Integrals*. Oxford University Press, London, UK, 2nd edition, 1948.

Todd, Neil P McAngus, Rosengren, Sally M, and Colebatch, James G. Tuning and sensitivity of the human vestibular system to low-frequency vibration. *Neuroscience Letters*, 444(1):36–41, 2008.

Tomilin, K. A. Natural systems of units. To the centenary anniversary of the Planck system. In *Proceedings Of The XXII Workshop On High Energy Physics And Field Theory*, 1999.

Tonomura, Akira, Osakabe, Nobuyuki, Matsuda, Tsuyoshi, Kawasaki, Takeshi, Endo, Junji, Yano, Shinichiro, and Yamada, Hiroji. Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave. *Physical Review Letters*, 56(8):792, 1986.

Travers, Susan P and Norgren, Ralph. The time course of solitary nucleus gustatory responses: Influence of stimulus and site of application. *Chemical Senses*, 14(1):55–74, 1989.

Trevarthen, Colwyn B. Two mechanisms of vision in primates. *Psychologische Forschung*, 31(4): 299–337, 1968.

Triarhou, Lazaros C and Verina, Tatyana. The musical centers of the brain: Vladimir E. Larionov (1857–1929) and the functional neuroanatomy of auditory perception. *Journal of Chemical Neuroanatomy*, 77:143–160, 2016.

Turin, Luca. A spectroscopic mechanism for primary olfactory reception. *Chemical Senses*, 21(6): 773–791, 1996.

Turker, Sabrina, Kuhnke, Philipp, Eickhoff, Simon B, Caspers, Svenja, and Hartwigsen, Gesa. Cortical, subcortical, and cerebellar contributions to language processing: A meta-analytic review of 403 neuroimaging experiments. *Psychological Bulletin*, 149(11–12):699–723, 2023.

Uchimura, Motoaki, Kumano, Hironori, and Kitazawa, Shigeru. Neural transformation from retinotopic to background-centric coordinates in the macaque precuneus. *Journal of Neuroscience*, 44 (48), 2024.

Vaidman, Lev. Role of potentials in the Aharonov-Bohm effect. *Physical Review A*, 86(4):040101, 2012.

Vaidman, Lev. Reply to "Comment on 'Role of potentials in the Aharonov-Bohm effect"'. *Physical Review A*, 92(2):026102, 2015.

Vaidman, Lev. Weak value controversy. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 375(2106):20160395, 2017.

Vaidman, Lev. Derivations of the Born rule. In Hemmo, Meir and Shenker, Orly, editors, *Quantum, probability, logic: The work and influence of Itamar Pitowsky*, pages 567–584. Springer Nature, Cham, Switzerland, 2020.

Vakman, DE and Vaĭnshteĭn, LA. Amplitude, phase, frequency—fundamental concepts of oscillation theory. *Soviet Physics Uspekhi*, 20(12):1002, 1977.

Van Der Pol, Balthasar. Frequency modulation. *Proceedings of the Institute of Radio Engineers*, 18 (7):1194–1205, 1930.

Van der Pol, Balthasar. The fundamental principles of frequency modulation. *Proc. IEE*, 93(111): 153–158, 1946.

Van Nes, FL, Koenderink, JJ, Nas, H, and Bouman, MA. Spatiotemporal modulation transfer in the human eye. *Journal of the Optical Society of America*, 57(9):1082–1088, 1967.

Van Raamsdonk, Mark. Building up space–time with quantum entanglement. *International Journal of Modern Physics D*, 19(14):2429–2435, 2010.

van Strien, Marij. Was physics ever deterministic? The historical basis of determinism and the image of classical physics. *The European Physical Journal H*, 46(1):8, 2021.

Vigran, Tor Erik. *Building Acoustics*. Taylor & Francis, New York, NY, 2009.

Ville, J. Theorie et application de la notion de signal analytique. *Câbles et Transmissions*, 2(1):61–74, 1948. Translated from French by I. Selin, "Theory and applications of the notion of complex signal," RAND Corporation Technical Report T-92, Santa Monica, CA, 1958.

Viterbi, A. J. Acquisition and tracking behavior of phase-locked loops. Technical Report External Publication No. 673, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, July 1959.

von Helmholtz, H. Theorie der Luftschwingungen in Röhren mit offenen Enden. *Journal für die reine und angewandte Mathematik*, 57:1–72, 1860.

von Neumann, John. *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, Princeton & Oxford, Princeton, NJ, 1932 / 2018. Translated from German by Robert T. Beyer. Edited by Nicholas A. Wheeler.

Voss, Richard P and Clarke, John. '1/f noise' in music and speech. *Nature*, 258:317–318, 1975.

Wallace, David. Philosophy of quantum mechanics. In Rickles, Dean, editor, *The Ashgate Companion to Contemporary Philosophy of Physics*, pages 16–98. Routledge, New York, NY, 2008.

Wandell, Brian A, Dumoulin, Serge O, and Brewer, Alyssa A. Visual field maps in human cortex. *Neuron*, 56(2):366–383, 2007.

Wang, DeLiang. On ideal binary mask as the computational goal of auditory scene analysis. In Divenyi, P., editor, *Speech Separation by Humans and Machines*, pages 181–197. Kluwer Academic, Norwell, MA, 2005.

Watson, Helen. Biological membranes. *Essays in Biochemistry*, 59:43–69, 2015.

Weaver, Warren. Science and complexity. *American Scientist*, 36(4):536–544, 1948.

Weder, Ricardo. The electric Aharonov-Bohm effect. *Journal of Mathematical Physics*, 52(5), 2011.

Weinert, Friedel. *The Demons of Science: What They Can and Cannot Tell Us About Our World*. Springer International Publishing Switzerland, 2016.

Weisser, Adam. Treatise on hearing: The temporal auditory imaging theory inspired by optics and communication. *arXiv preprint arXiv:2111.04338*, 2021.

Welch, Peter. The use of fast Fourier transform for the estimation of power spectra: A method based on time averaging over short, modified periodograms. *IEEE Transactions on Audio and Electroacoustics*, 15(2):70–73, 1967.

Welchman, Andrew E. The human brain in depth: How we see in 3D. *Annual Review of Vision Science*, 2:345–376, 2016.

Wells, H. G. *The Time Machine*. William Heinemann, London, 1895.

Weyl, Hermann. *The Theory of Groups and Quantum Mechanics*. Dover Publications, Inc., NY, 1928 / 1950. Translated from the 2nd German edition by H.P. Robertson; Originally published in 1928.

Whitham, G. B. *Linear and Nonlinear Waves.* John Wiley & Sons, Inc., New York, NY, 1999. First published in 1974.

Wiener, Norbert. Generalized harmonic analysis. *Acta Mathematica*, 55:117–258, 1930.

Wiener, Norbert. *I Am a Mathematician: The later Life of a Prodigy.* The M.I.T. Press, Cambridge, MA, 1956.

Wiener, Norbert and Struik, DJ. The fifth dimension in relativistic quantum theory. *Proceedings of the National Academy of Sciences of the United States of America*, 14(3):262, 1928.

Wigner, Eugene. On the quantum correction for thermodynamic equilibrium. *Physical Review*, 40 (5):749–759, 1932.

Wigner, Eugene P. Remarks on the body-mind question. In Wheeler, John Archibald and Zurek, Wojciech Hubert, editors, *Quantum Theory and Measurement*, pages 168–181. Princeton, New Jersey, 1961 / 1983. Originally published in The Scientist Speculates, I. J. Good, ed., pp. 284–302, Heinemann, London (1961).

Wilbraham, Henry. On a certain periodic function. *The Cambridge and Dublin Mathematical Journal*, III:198–201, 1848.

Williams, Kathy S and Simon, Chris. The ecology, behavior, and evolution of periodical cicadas. *Annual Review of Entomology*, 40(1):269–295, 1995.

Witten, Edward. Reflections on the fate of spacetime. *Physics Today*, 49(4):24–30, 1996.

Woit, Peter. Is spacetime really doomed? *International Journal of Modern Physics D*, 31(14): 2242005, 2022.

Wright, Robert H. Odor and molecular vibration: Neural coding of olfactory information. *Journal of Theoretical Biology*, 64(3):473–502, 1977.

Wurm, Lee H, Legge, Gordon E, Isenberg, Lisa M, and Luebker, Andrew. Color improves object recognition in normal and low vision. *Journal of Experimental Psychology: Human perception and performance*, 19(4):899, 1993.

Young, Thomas. I. the Bakerian Lecture. Experiments and calculations relative to physical optics. *Philosophical Transactions of the Royal Society of London*, (94):1–16, 1804.

Yu, Fei, Li, Lixiang, Tang, Qiang, Cai, Shuo, Song, Yun, and Xu, Quan. A survey on true random number generators based on chaos. *Discrete Dynamics in Nature and Society*, 2019(1):2545123, 2019.

Zeh, H Dieter. On the interpretation of measurement in quantum theory. *Foundations of Physics*, 1(1):69–76, 1970.

Zhou, Zhi-Yuan, Zhu, Zhi-Han, Liu, Shi-Long, Li, Yin-Hai, Shi, Shuai, Ding, Dong-Sheng, Chen, Li-Xiang, Gao, Wei, Guo, Guang-Can, and Shi, Bao-Sen. Quantum twisted double-slits experiments: Confirming wavefunctions' physical reality. *Science Bulletin*, 62(17):1185–1192, 2017.

Zukowski, Marek, Zeilinger, Anton, Horne, M, and Ekert, Artur. "Event-ready-detectors" Bell experiment via entanglement swapping. *Physical Review Letters*, 71(26), 1993.

Zurek, Wojciech H. Decoherence and the transition from quantum to classical—revisited. *Los Alamos Science*, 27:86–109, 2002.